1. (20 points) Give the **General Solution** of the following Differential Equation:

$$t^3 y' + 4t^2 y = e^{-t}\,, \qquad t > 0\,.$$

$$y' + \frac{4}{t} = \frac{1}{t^3} e^{-t}$$

$$IF = \exp \int \frac{4}{t}\, dt = t^4$$

$$\frac{d}{dt}\left(t^4 y\right) = te^{-t}$$

$$t^4 y = -te^{-t} \quad e^{-t} + C$$

$$y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$$

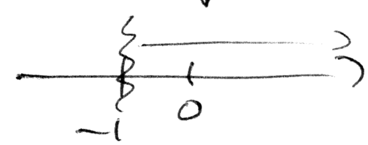
2. Consider the following Initial Value Problem:

$$\begin{cases} (1+t)y' + y = 2\cos t\,, \\ y(0) = 3\,. \end{cases}$$

a. (10 points) Determine (**without solving the equation**) the interval on which the solution of this Initial Value Problem is certain to exist.

$1+t = 0$ @ $t = -1$

$\left.\begin{matrix} 1 \\ 2\cos t \end{matrix}\right\}$ discont. never



$$\boxed{t > -1}$$

b. (10 points) Give the **General Solution** of the Differential Equation $(1+t)y' + y = 2\cos t$.

$$y' + \frac{1}{1+t}\, y = \frac{2\cos t}{1+t}$$

$$IF = \exp \int \frac{dt}{1+t} = 1+t$$

$$\frac{d}{dt}\left((1+t)y\right) = 2\cos t$$

$$(1+t)y = 2\sin t + C$$

$$\boxed{y = \frac{2\sin t + C}{1+t}}$$

c. (5 points) Solve the Initial Value Problem $\begin{cases} (1+t)y' + y = 2\cos t\,, \\ y(0) = 3\,. \end{cases}$

$$y = \frac{2\sin t + C}{1+t}$$

$$3 = \frac{2\cdot 0 + C}{1+0} = C$$

$$\Rightarrow \boxed{y = \frac{2\sin t + 3}{1+t}}$$

d. (5 points) Determine the behaviour of the solution of the Initial Value Problem as t goes to infinity.

$$\lim_{t\to\infty} y = 0$$

$$|2\sin t| < 2$$

$$\frac{5}{\infty} = 0$$

3. Determine whether or not each of the following equations is exact (5 points each):

a. $x^3 + \dfrac{y}{x} + \left(y^2 + \ln x\right) y' = 0$

$$\frac{\partial}{\partial y}\left(x^3 + y/x\right) = 1/x$$

$$\frac{\partial}{\partial x}\left(y^2 + \log x\right) = \frac{1}{x}$$

yes, exact

b. $(e^x \sin y + \tan y) + (e^x \cos y + x)\, y' = 0$

$$\frac{\partial}{\partial y}\left(e^x \sin y + \tan y\right) = e^x \cos y + \sec^2 y$$

$$\neq$$

$$\frac{\partial}{\partial x}\left(e^x \cos y + x\right) = e^x \cos y + 1$$

no, not exact

c. $\cos x + \ln y + \left(\dfrac{x}{y} + e^y\right) y' = 0$

$$\frac{\partial}{\partial y}\left(\cos x + \log y\right) = \frac{1}{y}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{y} + e^y\right) = \frac{1}{y}$$

yes, exact

5. (15 points) Solve the following Initial Value Problem (~~Hint: First rewrite the Differential Equation as a Separable Equation~~):

$$\begin{cases} y' = 2xy^2 + 3x^2y^2 \,, \\ y(1) = -1 \,. \end{cases}$$

$$\frac{y'}{y^2} = 2x + 3x^2 \quad \underline{\text{or}} \quad y = 0$$

$$-\frac{1}{y} = x^2 + x^3 + C \quad \text{or} \quad y = 0$$

$$y = -\frac{1}{x^3 + x^2 + C} \quad \underline{\text{or}} \quad y = 0$$

$$\underbrace{-1 = y(1)}_{(y \neq 0)} = -\frac{1}{1+1+C} \Rightarrow C = -1$$

$$\boxed{y = -\frac{1}{x^3 + x^2 - 1}}$$

5. Consider the following Initial Value Problem:

$$\begin{cases} 2y(\sin x)(\cos x) + (4y + \sin^2 x)y' = 0\,, \\ y(0) = 1\,. \end{cases}$$

a. (5 points) Prove that the Differential Equation is exact.

$$\frac{\partial}{\partial y}\left(2y \sin x \cos x\right) = 2 \sin x \cos x$$

$$\frac{\partial}{\partial x}\left(4y + \sin^2 x\right) = 2 \sin x \cos x$$

b. (15 points) Solve the Initial Value Problem.

$$\frac{\partial F}{\partial y} = 4y + \sin^2 x$$

$$F = 2y^2 + y \sin^2 x + C(x)$$

$$\frac{\partial F}{\partial x} = 2y \sin x \cos x + C'(x)$$

$$C'(x) = 0$$

$$C(x) = D$$

$$\frac{\partial F}{\partial x} = 2y \sin x \cos x$$

$$F = y \sin^2 x + C(y)$$

$$\frac{\partial F}{\partial y} = \sin^2 x + C'(y)$$

$$C'(y) = 4y$$

$$C(y) = 2y^2 + D$$

$$\Rightarrow \boxed{y \sin^2 x + 2y^2 = E}$$

$y(0) = 1$

$\Rightarrow E = 2$

& choose the "+" root so that $y = \ldots$

$$y = \frac{-\sin^2 x \pm \sqrt{\sin^4 x + 8E}}{4}$$