1. Give the general solution of the following equations (5 pts each):

a. $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$(r+1)^2 + 1 = 0$

$r = -1 \pm i$

$$\boxed{y = Ce^{-t} \sin t + De^{-t} \cos t}$$

b. $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0 \qquad r = 2$ (twice)

$$\boxed{y = Ce^{2t} + Dte^{2t}}$$

c. $y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0$

$r = \pm i$ (twice)

$$\boxed{y = C\sin t + D\cos t + Et\sin t + Ft\cos t}$$

2. (10 pts) Find the solution of the Initial Value Problem

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 2 \\ y'(0) = -3. \end{cases}$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$y = Ce^{3t} + De^{-t}$$

$$y' = 3Ce^{3t} - De^{-t}$$

$$2 = C + D$$

$$-3 = 3C - D$$

$$C = -1/4 \qquad D = 2\,1/4$$

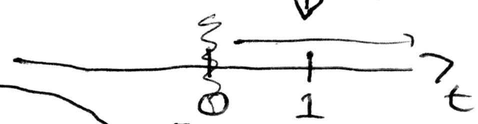
$$\boxed{y = -\frac{1}{4}e^{3t} + \frac{9}{4}e^{-t}}$$

3. Consider the Initial Value Problem: $\begin{cases} t^2y'' - ty' + y = 0 \\ y(1) = 7 \\ y'(1) = 2 \end{cases}$

a. (5 pts) Determine the longest interval in which the Initial Value Problem is certain to have a unique solution.

$t^2 = 0$ @ $t = 0$

$-t$ & 1 are never discont.



$\boxed{t > 0}$

b. (5 pts) Show that $y_1(t) = t$ and $y_2(t) = t\ln t$ are solutions of the differential equation

$$t^2y'' - ty' + y = 0$$

$$\begin{aligned} 1\,(y_1 &= t) \\ -t\,(y_1' &= 1) \\ +\ t^2\,(y_1'' &= 0) \\ \hline 0 &= t - t \ \checkmark \end{aligned}$$

$$\begin{aligned} 1\,(y_2 &= t\log t) \\ -t\,(y_2' &= \log t + 1) \\ +\ t^2\,(y_2'' &= 1/t) \\ \hline 0 &= t - t\log t - t + t\log t \ \checkmark \end{aligned}$$

c. (10 pts) Determine whether $y_1(t)$ and $y_2(t)$ are linearly dependent or linearly independent.

$$\begin{vmatrix} t & t\log t \\ 1 & \log t + 1 \end{vmatrix} = t\log t + t - t\log t = t$$

The Wronskian of t & $t\log t$ is not always 0

so $\boxed{\text{yes, they are independent.}}$

d. (5 pts) Using **3.b** and **3.c**, give the general solution of the equation $t^2y'' - ty' + y = 0$.

$$\boxed{y = Ct + Dt\log t}$$

4. Consider the equation

$$y'' + 4y' + 4y = e^{-2t} + \sin 2t$$

a. (5 pts) Give the **general solution** of the corresponding **homogeneous** equation

$$y'' + 4y' + 4y = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \qquad r = -2 \quad (\text{twice})$$

$$\boxed{y_H = Ce^{-2t} + Dte^{-2t}}$$

b. (15 pts) Find a **particular solution** of the **nonhomogeneous equation**

$$y'' + 4y' + 4y = e^{-2t} + \sin 2t$$

$$4\,(\; y_I = (Ae^{-2t})t^2$$

$$4\,(\; y_I' = 2tAe^{-2t} - 2t^2Ae^{-2t}$$

$$+\;1\,(\; y_I'' = 2Ae^{-2t} - 8tAe^{-2t} + 4t^2Ae^{-2t}$$

$$e^{-2t} = 2Ae^{-2t}$$

$$1 = 2A$$

$$A = 1/2$$

$$y_{II} = A\sin(2t) + B\cos(2t)$$

$$y_{II}' = -2B\sin(2t) + 2A\cos(2t)$$

$$y_{II}'' = -4A\sin(2t) - 4B\cos(2t)$$

$$\sin 2t = (-8B)\sin(2t) + 8A\cos 2t$$

$$1 = -8B \qquad B = -1/8$$

$$0 = 8A \qquad A = 0$$

$$\boxed{y_P} = y_I + y_{II}$$

$$= \boxed{\frac{1}{2}t^2e^{-2t} - \frac{1}{8}\cos(2t)}$$

c. (5 pts) Using **4.a** and **4.b**, give the **general solution** of the **nonhomogeneous equation**

$$y'' + 4y' + 4y = e^{-2t} + \sin 2t$$

$y = y_H + y_P = y_H + y_I + y_{II}$

$$\boxed{y = Ce^{-2t} + Dte^{-2t} + \frac{1}{2}t^2e^{-2t} - \frac{1}{8}\cos 2t}$$

5. (5 pts) Determine the radius of convergence of the following power series:

$$\sum_{n=0}^{+\infty} \frac{n}{4^n}(x-2)^n$$

Ratio test: $\lim_{n\to\infty} \left| \frac{(n+1)/4^{n+1} \cdot (x-2)^{n+1}}{n/4^n \cdot (x-2)^n} \right| < 1 \Rightarrow$ convergence

$$\|$$

$$\lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{4} \cdot (x-2) \right| = \frac{1}{4}|x-2|$$

$\frac{1}{4}|x-2| < 1 \to |x-2| < 4 \to \boxed{ROC = 4}$

6. Consider the equation

$$y'' + 4y = \sec(2t)$$

a. (5 pts) Give the **general solution** of the corresponding **homogeneous equation**: $y'' + 4y = 0$.

$r^2 + 4 = 0$

$r = \pm 2i$

$$\boxed{y_H = C \sin 2t + D \cos 2t}$$

b. (15 pts) Find a **particular solution** of the **non-homogeneous equation**, using the Method of Variation of Parameters.

$$y_I = u_1 \sin 2t + u_2 \cos 2t$$

$$0 = u_1' \sin 2t + u_2' \cos 2t$$

$$\sec 2t = u_1'(2\cos 2t) + u_2'(-2\sin 2t)$$

$$u_1' = -\frac{\cos 2t \cdot \sec 2t}{W(\sin 2t, \cos 2t)} \qquad u_2' = \frac{\sin 2t \cdot \sec 2t}{W(\sin 2t, \cos 2t)}$$

$$W(\sin 2t, \cos 2t) = \begin{vmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{vmatrix}$$

$$= -2\sin^2 2t - 2\cos^2 2t = -2$$

$$u_1' = \frac{1}{2} \quad \& \quad u_2' = -\frac{1}{2}\tan 2t = -\frac{1}{2} \cdot \frac{\sin 2t}{\cos 2t}$$

(next page)

$$u_1 = \frac{1}{2} t \quad \& \quad u_2 = \frac{1}{4} \log|\cos 2t|$$

$$\boxed{y_I = \frac{1}{2} t \sin 2t + \frac{1}{4}\left(\log|\cos 2t|\right)\left(\cos 2t\right)}$$