§2.2

③ $y' + y^2 \sin x = 0$

$$\frac{dy}{dx} = -y^2 \sin x$$

$y \neq 0$ $\underline{\underline{\text{or}}}$ $\boxed{y=0}$

($y'=0$ so DE is ok)

$$\frac{dy}{y^2} = -\sin x \, dx$$

$$-y^{-1} = \cos x + C$$

$$\boxed{y = \frac{-1}{\cos x + C}}$$ (so long as $\cos x + C \neq 0$)

⑥ $x y' = (1-y^2)^{1/2}$

($y \neq \pm 1$ & $x \neq 0$) $\underline{\text{OR}}$ $\boxed{y = \pm 1}$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x}$$

$$\sin^{-1} y = \log|x| + C \longrightarrow \boxed{y = \sin\left(\log|x| + C\right)}$$

⑦ $$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^{y}}$$

$$(y + e^{y})\,dy = (x - e^{-x})\,dx$$

$$\boxed{y^2/2 + e^{y} = x^2/2 + e^{x} + C}$$

⑩ $y' = (1-2x)/y \qquad y(1) = -2$

$$y\,dy = (1-2x)\,dx$$

$$y^2/2 = x - x^2 + C$$

$$y = \pm\sqrt{2(x - x^2 + C)}$$

$$-2 = y(1) = \pm\sqrt{2(1 - 1^2 + C)} \implies C = 2 \text{ \& pick the "$-$" root}$$

$$\boxed{y = -\sqrt{2(x - x^2 + 2)}}$$

This is def'd and differentiable where the radicand is ~~positive~~ ~~non negative~~ positive:

$$x - x^2 + 2 > 0 \implies (2-x)(1+x) > 0 \implies \boxed{-1 < x < 2}$$

(12) $\frac{dr}{d\theta} = \frac{r^2}{\theta}$ $\quad r(1) = 2$

$r \neq 0$ <u>OR</u> $r = 0$

$\frac{dr}{r^2} = \frac{d\theta}{\theta}$ this doesn't fit the IC

$-\frac{1}{r} = \log|\theta| + C$

$r = \frac{-1}{\log|\theta| + C}$ $\quad r(1) = 2 \Rightarrow C = -1/2$

$$\boxed{r = \frac{-1}{\log|\theta| - 1/2} = \frac{2}{1 - 2\log|\theta|} \quad \text{for } 0 < \theta < e^{1/2}}$$

This has discontinuities at $\theta = 0$ & $\theta = e^{1/2}$

since $0 < 1 < e^{1/2}$

$\uparrow$ from IC

(23) $y' = 2y^2 + xy^2$ $\quad y(0) = 1$

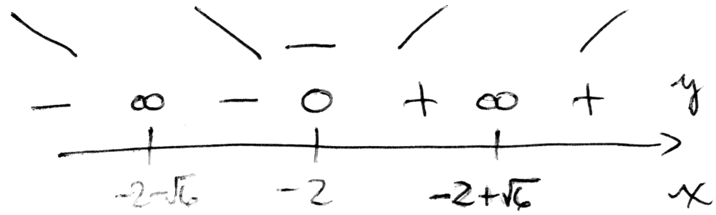
$\frac{dy}{y^2} = (2+x)\,dx$ $\quad$ ($y = 0$ doesn't fit the IC)

$-\frac{1}{y} = 2x + x^2/2 + C$ $\quad y(0) = 1 \Rightarrow C = -1$

$$y = -\frac{1}{2x + x^2/2 - 1}$$

(23) (cont)

$$y' = \frac{2+x}{\left(2x + x^2/2 - 1\right)^2}$$



$\Rightarrow$ y has a min @ $x = -2$

(25) $$y' = \frac{2\cos 2x}{3+2y} \qquad y(0) = -1$$

$$(3+2y)\,dy = 2\cos 2x\,dx$$

$$3y + y^2 = \sin 2x + C \qquad y(0) = -1 \Rightarrow C = -2$$

$$3y + y^2 = \sin 2x - 2$$

$$\left(y + 3/2\right)^2 = \sin 2x + 1/4$$

$$y = -\frac{3}{2} + \sqrt{\sin 2x + 1/4}$$

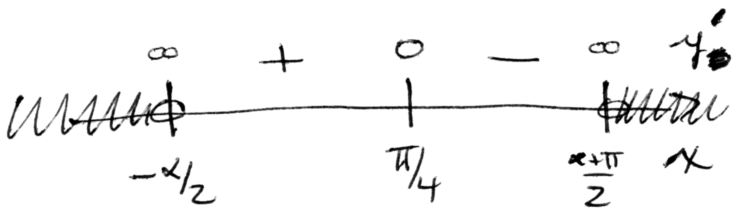
$$y' = \frac{2\cos 2x}{2\sqrt{\sin 2x + 1/4}}$$

(25) (cont)

y is only defined and differentiable where $(\sin 2x + 1/4) > 0$

This is multiple disconnected intervals. We want the interval which contains the intial data @ $x=0$.

If $\alpha = \sin^{-1}(1/4)$, this is $\left(-\alpha/2, \frac{\alpha+\pi}{2}\right)$.

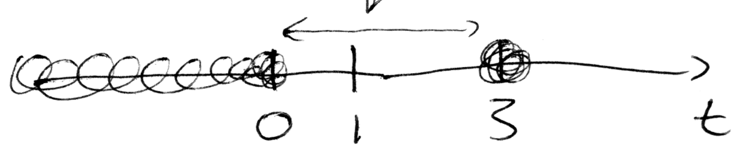


$\Rightarrow$ y has a max @ $x = \pi/4$

§2.4

(1) $\underbrace{(t-3)}_{\downarrow} y' + \underbrace{(\log t)}_{\text{continuous for } t>0} y = \underbrace{2t}_{\text{cont. everywhere}} \quad y(1) = 2$

continuous & not zero for $t \neq 3$



$\Rightarrow$ theorem 2.4.1 guarantees a unique sol'n for $0 < t < 3$.

(4) $(4-t^2)\,y' + 2t\,y = 3t^2 \qquad y(-3)=1$

$(4-t^2)$: cont. & nonzero for all t but $t=\pm 2$

$2t$, $3t^2$: $\to$ cont. everywhere

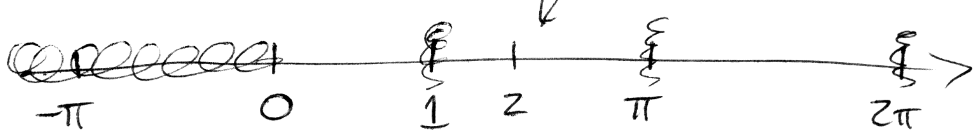
$\Rightarrow$ Thm 2.4.1 guar. a unique sol'n for $t<-3$.

(6) $(\log t)\,y' + 1\,y = \cot t \qquad y(2)=3$

$(\log t)$: cont. & nonzero for $0<t<1$ and $1<t$

1: cont. for all t

$\cot t$: cont for all t but $t=n\pi$ (integer mult. of π)



$\Rightarrow$ Thm 2.4.1 guar. a unique sol'n for $1<t<\pi$

(8) $y' = (1 - t^2 - y^2)^{1/2}$

$f(t,y) = (1 - t^2 - y^2)^{1/2}$ $\longrightarrow$ continuous for all t & y where $1 - t^2 - y^2 \geq 0$

$\Rightarrow t^2 + y^2 \leq 1$

$\frac{\partial f}{\partial y}(t,y) = \frac{1}{2}(1 - t^2 - y^2)^{-1/2}(-2y)$ $\rightarrow$ continuous for all t & y

where $1 - t^2 - y^2 \geq 0$

and $1 - t^2 - y^2 \neq 0$

together, we want $\boxed{t^2 + y^2 < 1}$ (circle of radius 1 centered @ origin)

(10) $y' = (t^2 + y^2)^{3/2}$

$f(t,y) = (t^2 + y^2)^{3/2}$

$\frac{\partial f}{\partial y}(t,y) = \frac{3}{2}(t^2 + y^2)^{1/2}(2y)$

$\rightarrow$ continuous for $t^2 + y^2 \geq 0$

$\Rightarrow$ $\boxed{\text{everywhere}}$

(13) $y' = -4t/y \qquad y(0) = y_0$

$y y' = -4t$

$y^2/2 = -2t^2 + C \longrightarrow C = y_0^2/2$

$y = \pm\sqrt{2(C - 2t^2)}$

$$\boxed{y = \operatorname{sign}(y_0) \cdot \sqrt{2\left(y_0^2/2 - 2t^2\right)}}$$

we need $2\left(y_0^2/2 - 2t^2\right) > 0$ for y to be continuous & differentiable. so $\boxed{|t| < |y_0/2|}$

(14) $y' = 2t y^2 \qquad y(0) = y_0$

$\frac{y'}{y^2} = 2t$ if $y \neq 0$ or $\boxed{y = 0}$ (only if $y_0 = 0$)

$-\frac{1}{y} = t^2 + C$

$y = \frac{-1}{t^2 + C} \qquad C = -1/y_0$ (only if $y_0 \neq 0$)

$$\boxed{y = \frac{-1}{t^2 - 1/y_0}}$$

(14) (cont).

Case

$y_0 = 0$ $\quad y = 0$ $\quad$ for all t

$y_0 < 0$ $\quad y = -\dfrac{1}{t^2 - 1/y_0}$ $\quad$ for all t

$y_0 > 0$ $\quad y = -\dfrac{1}{t^2 - 1/y_0}$ $\quad$ for $|t| < \sqrt{1/y_0}$

(25) $y = y_1$ is a sol'n of

$$y' + py = 0 \quad (\mathrm{I})$$

$y = y_2$ is a sol'n of ~~■~~

$$y' + py = g \quad (\mathrm{II})$$

Verify that $y = y_1 + y_2$ is a sol'n of (II):

$$(y_1 + y_2)' + p(y_1 + y_2) \stackrel{?}{=} g$$

$$y_1' + y_2' + py_1 + py_2 \stackrel{?}{=} g$$

$$y_1' + py_1 + y_2' + py_2 \stackrel{?}{=} g$$

$$\underbrace{y_1' + py_1}_{\downarrow} + \underbrace{y_2' + py_2}_{\downarrow}$$

$$0 + g = g \quad \checkmark$$