§2.6

① $(2x+3) + (2y-2)\,y' = 0$

$$(2x+3)\,dx + (2y-2)\,dy = 0$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0 = 0 = \frac{\partial^2 F}{\partial y \partial x} \quad \checkmark \quad \text{(is exact)}$$

$$\frac{\partial F}{\partial x} = 2x+3$$

$$\Rightarrow F = x^2 + 3x + C(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = C'(y)$$

$$\Rightarrow 2y-2 = C'(y)$$

$$\Rightarrow y^2 - 2y + D = C(y)$$

$$\Rightarrow F = x^2 + 3x + y^2 - 2y + D$$

sol'n to DE given by $F = \text{const}$

so $\boxed{x^2 + 3x + y^2 - 2y = \text{const}}$

(or $y = 1 \pm \sqrt{\text{CONST} - 3x - x^2}$)

② $(2x+4y) + (2x-2y)y' = 0$

$$(2x+4y)\,dx + (2x-2y)\,dy = 0$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = 4 \neq 2 = \frac{\partial^2 F}{\partial y \partial x}$$ (is <u>not</u> exact)

⑧ $(e^x \sin y + 3y)\,dx - (3x - e^x \sin y)\,dy = 0$

$$(e^x \sin y + 3y)\,dx + (e^x \sin y - 3x)\,dy = 0$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = e^x \cos y + 3 \neq e^x \sin y - 3 = \frac{\partial^2 F}{\partial y \partial x}$$ (<u>not</u> exact)

⑩ $(y/x + 6x)\,dx + (\log x - 2)\,dy = 0 \qquad \forall x > 0$

$$\partial F = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{1}{x} = \frac{\partial^2 F}{\partial y \partial x}$$ (is exact)

(10) (cont)

$$\frac{\partial F}{\partial x} = \frac{y}{x} + 6x$$

$$F = y \log x + 3x^2 + C(y)$$

$$\frac{\partial F}{\partial y} = \log x + C'(y)$$

Also

$$\frac{\partial F}{\partial y} = \log x - 2$$

so

$$\log x - 2 = \log x + C'(y)$$

$$\Rightarrow \quad -2 = C'(y)$$

$$\Rightarrow \quad -2y + D = C(y)$$

$$\Rightarrow \quad F = y \log x + 3x^2 - 2y + D$$

soll'n to DE:

$$\boxed{y \log x + 3x^2 - 2y = \text{CONST}}$$

$$y = \frac{\text{CONST} - 3x^2}{\log x - 2} \qquad (0 < x < e^2)$$

(16) $$\left(ye^{2xy} + x\right)dx + bxe^{2xy}\,dy = 0$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

with b as a constant:

$$\frac{\partial^2 F}{\partial x\,\partial y} = e^{2xy} + 2xye^{2xy} = b\left(e^{2xy} + 2xye^{2xy}\right) = \frac{\partial^2 F}{\partial y\,\partial x}$$

$\Rightarrow$ the DE is exact only for $b=1$.

$$\frac{\partial F}{\partial x} = ye^{2xy} + x$$

$$F = \frac{1}{2}e^{2xy} + \frac{1}{2}x^2 + C(y)$$

$$\frac{\partial F}{\partial y} = xe^{2xy} + C'(y)$$

$$\frac{\partial F}{\partial y} = bxe^{2xy} = xe^{2xy} \quad \Rightarrow C'(y) = 0$$

$$\Rightarrow C(y) = D$$

$$\boxed{\frac{1}{2}e^{2xy} + \frac{1}{2}x^2 = \text{CONST}}$$

$$y = \frac{1}{2x}\log\left(\widetilde{\text{CONST}} - x^2\right)$$

(20) $$\underbrace{\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)}_{\frac{\partial F}{\partial x}}dx + \underbrace{\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)}_{\frac{\partial F}{\partial y}}dy = 0$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial^2 F}{\partial x\partial y} = \frac{y\cos y - \sin y}{y^2} \neq \frac{2}{y}\left(-e^{-x}\cos x - e^{-x}\sin y\right) = \frac{\partial^2 F}{\partial y\partial x}$$

This is not exact. Mult. the DE by ye^{x} gives:

$$\underbrace{\left(e^{x}\sin y - 2y\sin x\right)}_{\frac{\partial F}{\partial x}}dx + \underbrace{\left(e^{x}\cos y + 2\cos x\right)}_{\frac{\partial F}{\partial y}}dy = 0$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial^2 F}{\partial x\partial y} = +e^{x}\cos y - 2\sin x = \frac{\partial^2 F}{\partial y\partial x} \quad \text{(is exact)}$$

$$\frac{\partial F}{\partial x} = e^{x}\sin y - 2y\sin x$$

$$F = e^{x}\sin y + 2y\cos x + C(y)$$

$$\frac{\partial F}{\partial y} = e^{x}\cos y + 2\cos x + C'(y)$$

$$\frac{\partial F}{\partial y} = e^{x}\cos y + 2\cos x \implies C'(y) = 0$$

$$\implies C(y) = D$$

sol'n to DE: $\boxed{e^{x}\sin y + 2y\cos x = \text{CONST}}$

(21) $y\,dx + (2x - ye^y)\,dy = 0$

$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$

$1 \neq 2$ (not exact)

Mult. the DE by y gives

$y^2 dx + (2xy - y^2 e^y)\,dy = 0$

$2y = 2y$ ✓

$\frac{\partial F}{\partial x} = y^2$

$F = xy^2 + C(y)$

$\frac{\partial F}{\partial y} = 2xy + C'(y)$

$\frac{\partial F}{\partial y} = 2xy - y^2 e^y \Rightarrow C'(y) = -y^2 e^y$

$\Rightarrow C(y) = -y^2 e^y + 2ye^y - 2e^y + D$

$\Rightarrow$ sol'n to DE:

$$\boxed{xy^2 - y^2 e^y + 2ye^y - 2e^y = \text{CONST}}$$

(25) Well, we know it's not exact so let's not waste time

$$\mathrm{IF}\left(3x^2y + 2xy + y^3\right) dx + \mathrm{IF}(x^2+y^2)\, dy = 0$$

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial \mathrm{IF}}{\partial y}\left(3x^2y + 2xy + y^3\right) + \mathrm{IF}\left(3x^2 + 2x + 3y^2\right)$$

$$\parallel$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial \mathrm{IF}}{\partial x}\left(x^2+y^2\right) + \mathrm{IF}\,(2x)$$

If $\mathrm{IF} = \mathrm{IF}(x)$, then $\frac{\partial \mathrm{IF}}{\partial y} = 0$ and

$$\frac{d\,\mathrm{IF}}{dx}\left(x^2+y^2\right) = \mathrm{IF}\left(3x^2 + 3y^2\right)$$

$$\frac{d\,\mathrm{IF}}{dx} = 3\,\mathrm{IF} \Rightarrow \mathrm{IF} = e^{3x}$$

(NB: guessing $\mathrm{IF} = \mathrm{IF}(y)$ won't work)

$$\frac{\partial F}{\partial y} = e^{3x}\left(x^2 + y^2\right)$$

$$F = e^{3x}\left(x^2y + \frac{1}{3}y^3\right) + C(x)$$

$$\frac{\partial F}{\partial x} = 3e^{3x}\left(x^2y + \frac{1}{3}y^3\right) + e^{3x}(2xy) + C'(x)$$

$$\frac{\partial F}{\partial x} = e^{3x}\left(3x^2y + 2xy + y^3\right) \Rightarrow C'(x) = 0$$

$$C(x) = D$$

(25) (cont)

sol'n to DE: $\boxed{e^{3x}\left(x^2 y + \frac{1}{3} y^3\right) = \text{CONST}}$

(26) $y' = e^{2x} + y - 1$ (NB: This is linear, too)

$$\left(e^{2x} + y - 1\right) dx + (-1)\, dy = 0$$

$$IF\left(e^{2x} + y - 1\right) dx + IF(-1)\, dy = 0$$

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial IF}{\partial y}\left(e^{2x} + y - 1\right) + IF = -\frac{\partial IF}{\partial x} = \frac{\partial^2 F}{\partial y \partial x}$$

If $IF = IF(x)$, then $\frac{\partial IF}{\partial y} = 0$ & $IF = e^{-x}$

(NB: $IF = IF(y)$ won't work.)

$$\frac{\partial F}{\partial x} = e^{x} + (y-1)e^{-x}$$

$$F = e^{x} - (y-1)e^{-x} + C(y)$$

$$\frac{\partial F}{\partial y} = -e^{-x} + C'(y)$$

$$\frac{\partial F}{\partial y} = -e^{-x} \Rightarrow C'(y) = 0 \Rightarrow C(y) = \text{const}$$

sol'n to DE: $\boxed{e^{x} - (y-1)e^{-x} = \text{const}}$

(27) $dx + \left(x/y - \sin y \right) dy = 0$

$$IF\,dx + IF\left(x/y - \sin y \right) dy = 0$$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial IF}{\partial y} = \frac{\partial IF}{\partial x}\left(\frac{x}{y} - \sin y \right) + IF\left(\frac{1}{y} \right) = \frac{\partial^2 F}{\partial y \partial x}$$

If $IF = IF(y)$, then $\frac{dIF}{dy} = IF \cdot \frac{1}{y} \Rightarrow IF = y$

(NB: $IF = IF(x)$ won't work)

$$\frac{\partial F}{\partial x} = y$$

$$F = xy + C(y)$$

$$\left. \begin{aligned} \frac{\partial F}{\partial y} &= x + C'(y) \\ \frac{\partial F}{\partial y} &= x - y \sin y \end{aligned} \right\} \Rightarrow \begin{aligned} C'(y) &= -y \sin y \\ C(y) &= y \cos y - \sin y + D \end{aligned}$$

sol'n to DE is $\boxed{xy + y \cos y - \sin y = \text{const}}$