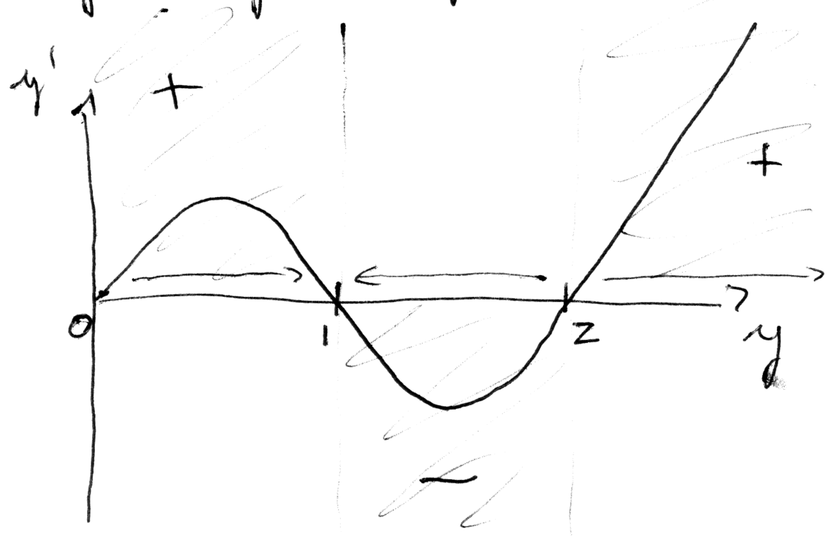
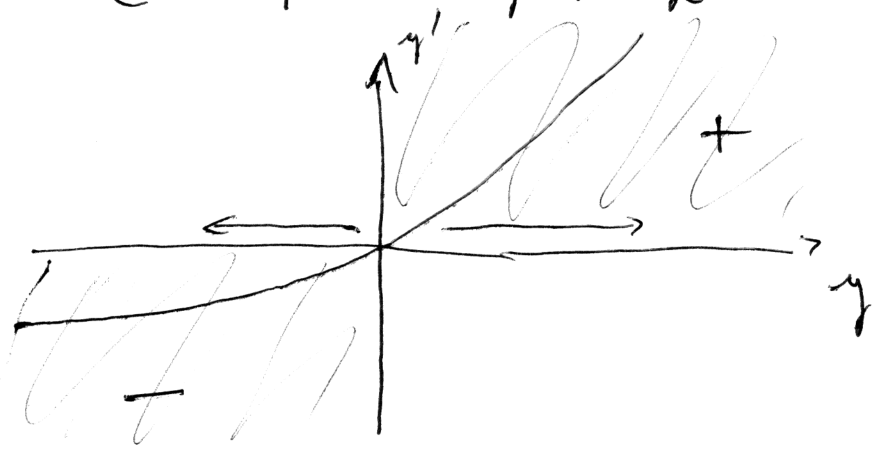
③ $y' = y(y-1)(y-2)$ $\qquad y(0) = y_0 \geq 0$



$y = 0, 1, 2$ are critical pts ($y' = 0$)

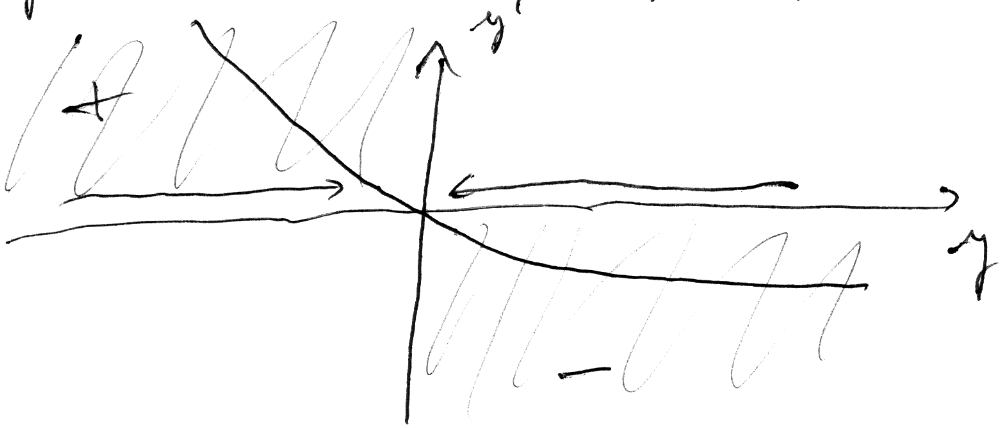
$y = 0, 2$ are unstable ; $y = 1$ is stable

④ $y' = e^{y} - 1$ $\qquad y(0) = y_0 \in \mathbb{R}$



$y = 0$ is an unstable critical pt

(5) $y' = e^{-y} - 1 \qquad y(0) = y_0 \in \mathbb{R}$



$y = 0$ is a stable critical pt

(7) $y' = k(1-y)^2 \quad w/\ k > 0$

(a) y is a critical pt

$$\Updownarrow$$

$$y' = 0 \quad \forall t$$

$$\Updownarrow$$

$$k(1-y)^2 = 0 \quad \forall t$$

$$\Updownarrow$$

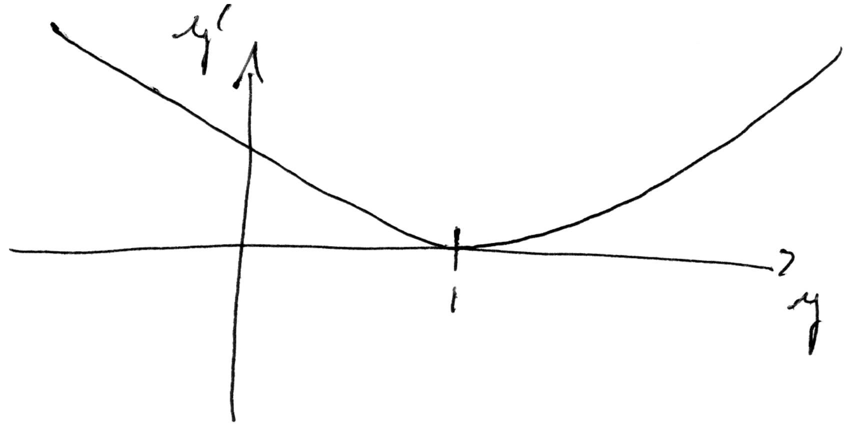
$$1 - y = 0 \quad \forall t$$

$$\Updownarrow$$

$$y = 1 \quad \forall t$$

(that the implications are bidirectional means $y = 1$ is a critical pt, is a sol'n, and is the only critical pt)

7 6



We repeat what is evident from the graph ($y' > 0$ $\forall y$ except $y' = 0$ for $y = 1$) using analysis:

$$y < 1 \Rightarrow 0 < 1 - y \Rightarrow 0 < (1-y)^2 \Rightarrow 0 < k(1-y)^2$$
$$\Rightarrow 0 < y'$$
$$\Rightarrow y \text{ is incr.}$$

$$y > 1 \Rightarrow 0 > 1 - y \Rightarrow 0 < (1-y)^2 \Rightarrow 0 < k(1-y^2)$$
$$\Rightarrow 0 < y'$$
$$\Rightarrow y \text{ is incr}$$

7 c $y' = k(1-y)^2$

$$\frac{dy}{(1-y)^2} = k\,dt \qquad \underline{\text{or}} \qquad y = 1$$

$$\frac{1}{1-y} = kt + C$$

$$y = 1 - \frac{1}{kt+C} \qquad \underline{\text{or}} \qquad y = 1$$

If $y(0) = y_0$, then

$y_0 \neq 1$

$$y_0 = 1 - \frac{1}{C}$$

$$C = \frac{1}{1-y_0}$$

$$y = 1 - \frac{1}{kt + \frac{1}{1-y_0}}$$

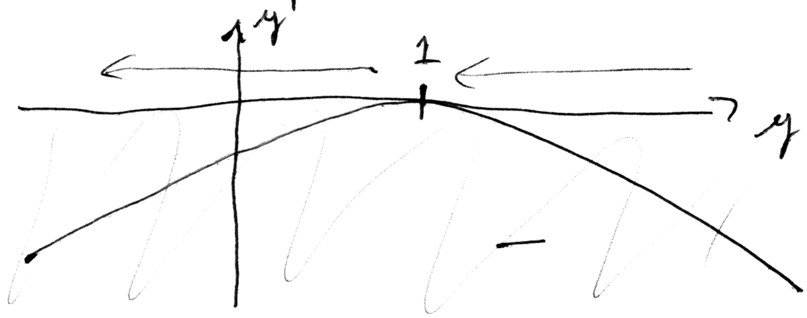
$y_0 = 1$

$y = 1$

critical pt (only one); verifies a

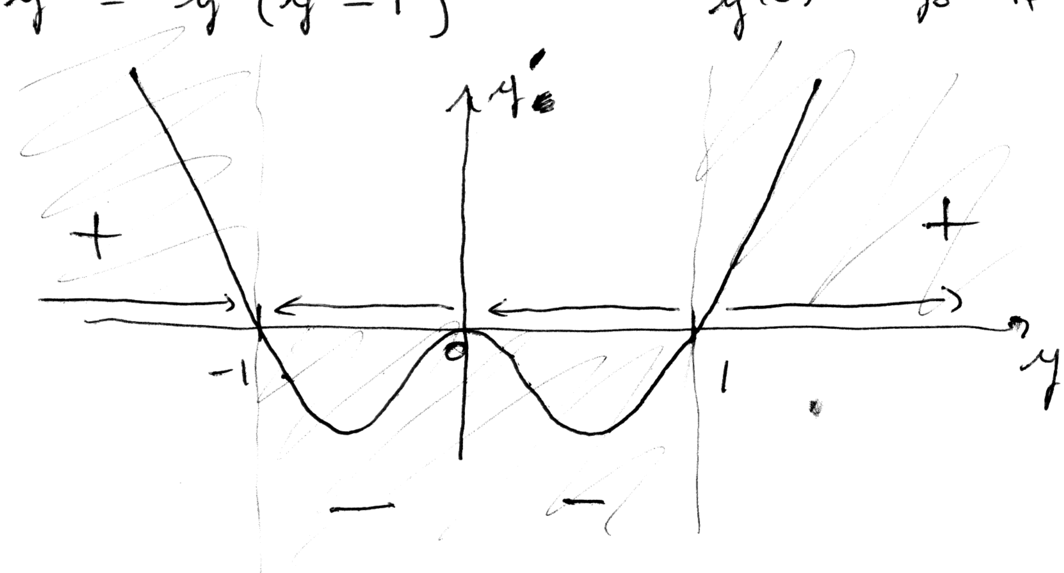
Since $k > 0$, kt is an increasing fcn of t (and so is $kt + \frac{1}{1-y_0}$). Thus $\frac{1}{kt+\frac{1}{1-y_0}}$ is a decreasing fcn of t, and $1 - \frac{1}{kt+\frac{1}{1-y_0}}$ is an increasing fcn of t (irrespective of y_0) confirming b.

(8) $y' = -k(y-1)^2 \qquad k>0 \qquad y(0) = y_0 \in \mathbb{R}$



$y=1$ is a semistable critical pt

(9) $y' = y^2(y^2-1) \qquad y(0) = y_0 \in \mathbb{R}$

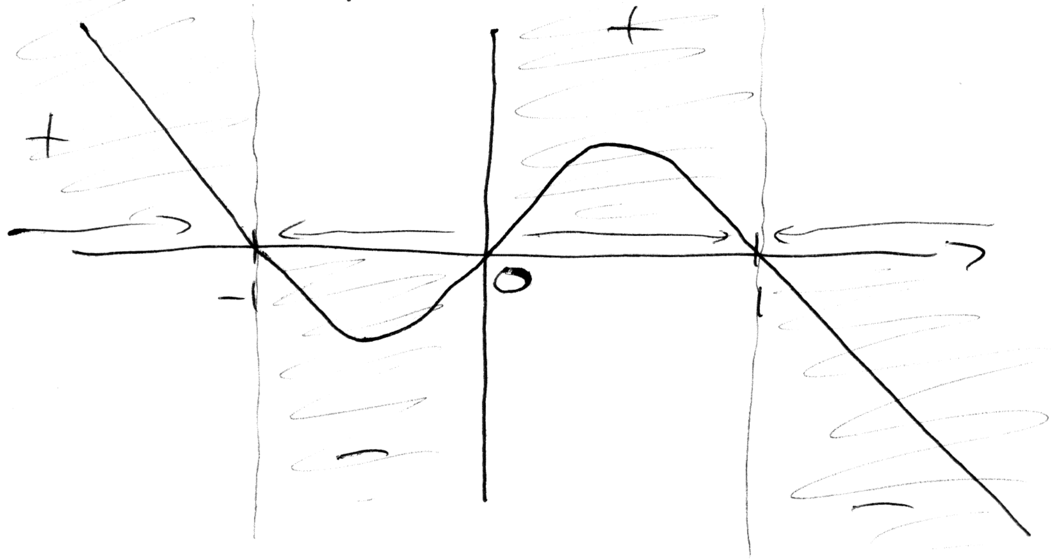


$y=-1$ is a stable critical pt

$y=0$ is a semistable critical pt

$y=1$ is an unstable critical pt

(10) $y' = y(1-y^2) \qquad y(0) = y_0 \in \mathbb{R}$



$y = \pm 1$ are stable crit pts

$y = 0$ is an unstable crit pt

(16) $y' = r y \log\left(K/y\right) \qquad r > 0 \quad K > 0$

(a) $f(y) = r y \log\left(K/y\right)$

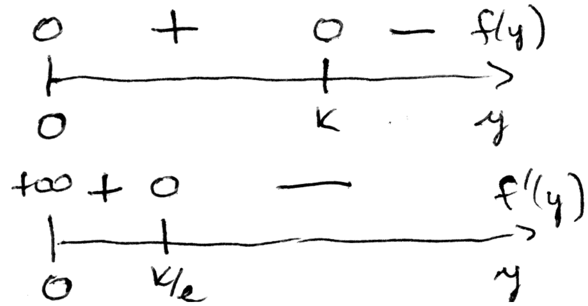
$f'(y) = r\left[\log\left(K/y\right) - 1\right]$

$f(y) = 0$ for $y = K$

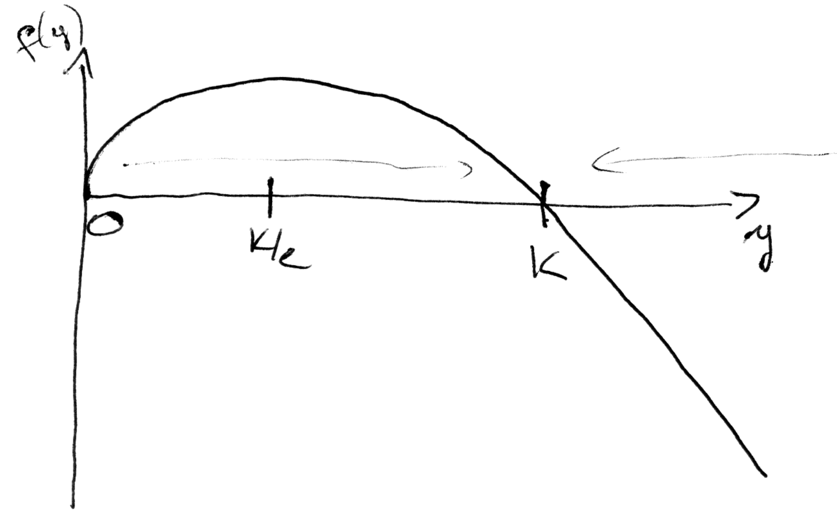
$\lim_{y \to 0^+} f(y) = 0$

$f'(y) = 0$ for $y = K/e$

$\lim_{y \to 0^+} f'(y) = +\infty$



(16) (a) (cont)



$y = 0$ is an unstable critical pt

$y = K$ is a stable critical pt

(b) NB: concavity of $y(t)$ <u>not</u> $f(y)$

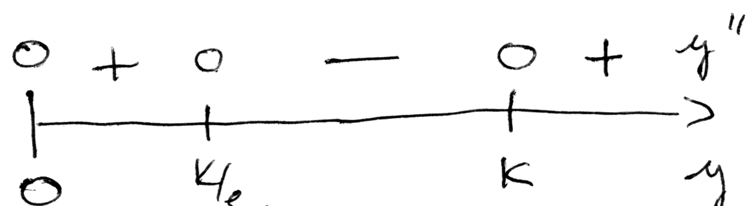
$$y' = ry \log(K/y)$$

$$y'' = r\left[y' \log(K/y) - y'\right]$$

$$= r\left[\log(K/y) - 1\right] y \log(K/y)$$

$$y'' = 0 \Rightarrow y = 0, K/e, K$$

↑ in the limit at least



y is concave up when $0 < y < K/e$ (or $K < y$) & concave down for $K/e < y < K$

16 c $y_G' = r y_G \log(K/y_G) = f_G(y_G)$ $\quad r, K > 0$

$y_L' = r y_L (1 - y_L/K) = f_L(y_L)$

The problem is to show that

$$f_G(y) \geq f_L(y) \quad \text{for} \quad 0 < y \leq K$$

If $0 < a \leq 1$, then

$$\frac{1}{a} \geq 1$$

$$-\frac{1}{a} \leq -1$$

Since $-\log 1 = 0 = 1 - 1$, then

$$-\log a \geq 1 - a \quad \text{for} \quad 0 < a \leq 1$$

Using $-\log a = \log(1/a)$ and $a = y/K$ we set

$$\log(K/y) \geq 1 - y/K \quad \text{for} \quad 0 < y/K \leq 1$$

$$(\text{or} \quad 0 < y \leq K)$$

Since $y > 0$ & $r > 0$, then multiplying by ry gives

$$\underbrace{ry \log(K/y)}_{= f_G(y)} \geq \underbrace{ry(1 - y/K)}_{= f_L(y)} \quad \text{for} \quad 0 < y \leq K$$