§3.3

Two functions f & g are dependent over a certain interval

$\Downarrow$

The Wronskian of f & g is 0 <u>everywhere</u> on that interval

(Pf. dependent $\Rightarrow$ $f = \underset{\substack{\uparrow \\ \text{constant}}}{C}g$ $\Rightarrow$ $f = Cg$ & $f' = Cg'$

$\Downarrow$

Wronskian is 0 everywhere $\Leftarrow$ $0 \underset{\text{"identically equal to"}}{\equiv} \begin{vmatrix} Cg & g \\ Cg' & g' \end{vmatrix} \underset{\text{"equal to"}}{=} \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

The contrapositive of the above is:

The Wronskian of f & g is not 0 <u>somewhere</u> on a certain interval

$\Downarrow$

f & g are independent on that interval

The converses of the above (converse = reverse implication) only come into play when you start thinking of f & g as sol'ns to some linear homogeneous DE (and not otherwise).

① $$w(f,g) = \begin{vmatrix} t^2+5t & t^2-5t \\ 2t+5 & 2t-5 \end{vmatrix}$$

$$= (t^2+5t)(2t-5) - (t^2-5t)(2t+5)$$

$$= 10t^2$$

so $W(f,g) \neq 0$ for some values of t

$\Rightarrow$ f, g are independent

~~ALT~~:

S'pose f & g are dependent. Then there is a constant C so that $f = Cg$ (for lots of t's — some interval of t's). Then

$$t^2 + 5t = C(t^2-5t)$$

$$\Rightarrow (1-C)t^2 + 5(1+C)t = 0$$

$$\Rightarrow t\left[(1-C)t + 5(1+C)\right] = 0$$

no matter what C is, this will have at most two roots (i.e., not many; not an interval of t's)

This is a contradiction to $f = Cg$ for lots of t's so f & g must be independent.

(4) $f(x) = e^{3x}$
$g(x) = e^{3(x-1)}$

$$\begin{vmatrix} e^{3x} & e^{3(x-1)} \\ 3e^{3x} & 3e^{3(x-1)} \end{vmatrix} = e^{3x} \cdot 3e^{3(x-1)} - e^{3(x-1)} \cdot 3e^{3x} = 0$$

$\uparrow$ for all x

$\Rightarrow$ f & g are dependent

(ALT: $f = e^3 \cdot g$ 'cuz $e^{3(x-1)} = e^{3x} \cdot e^{-3}$)

(6) $$\begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = -2t^{-1}$$ (never 0, at least where defined)

$\Rightarrow$ f & g are independent

(9) $t \sin^2 t$ is not 0 for some t (on any interval)

$\Rightarrow$ The two functions are independent (on any interval)

(15) $W = C\exp\left(-\int -\frac{t(t+2)}{t^2}\,dt\right)$ (use Thm 3.3.2; NB: This requires the STD form of the DE!)

$= C\exp\left(\int (1 + 2/t)\,dt\right)$

$= C\exp\left(t + 2\log t\right)$

$= Ct^2 e^t$

(16)

$$W = C \exp\left(-\int \frac{\sin t}{\cos t}\, dt\right)$$

$$= C \exp\left(-\left(-\log|\cos t|\right)\right)$$

$$= C \cos t$$

§3.5

(6) $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$$y = Ce^{3t} + Dte^{3t}$$

(11) $9y'' - 12y' + 4y = 0$ $\quad y(0) = 2 \quad y'(0) = -1$

$9r^2 - 4r + 4 = 0$

$(3r-2)^2 = 0$

$$y = Ce^{2/3t} + Dte^{2/3t}$$

$$y' = \frac{2}{3}Ce^{2/3t} + D\left(e^{2/3t} + \frac{2}{3}te^{2/3t}\right)$$

4 (cont)

$$2 = C \cdot 1 + D \cdot 0$$

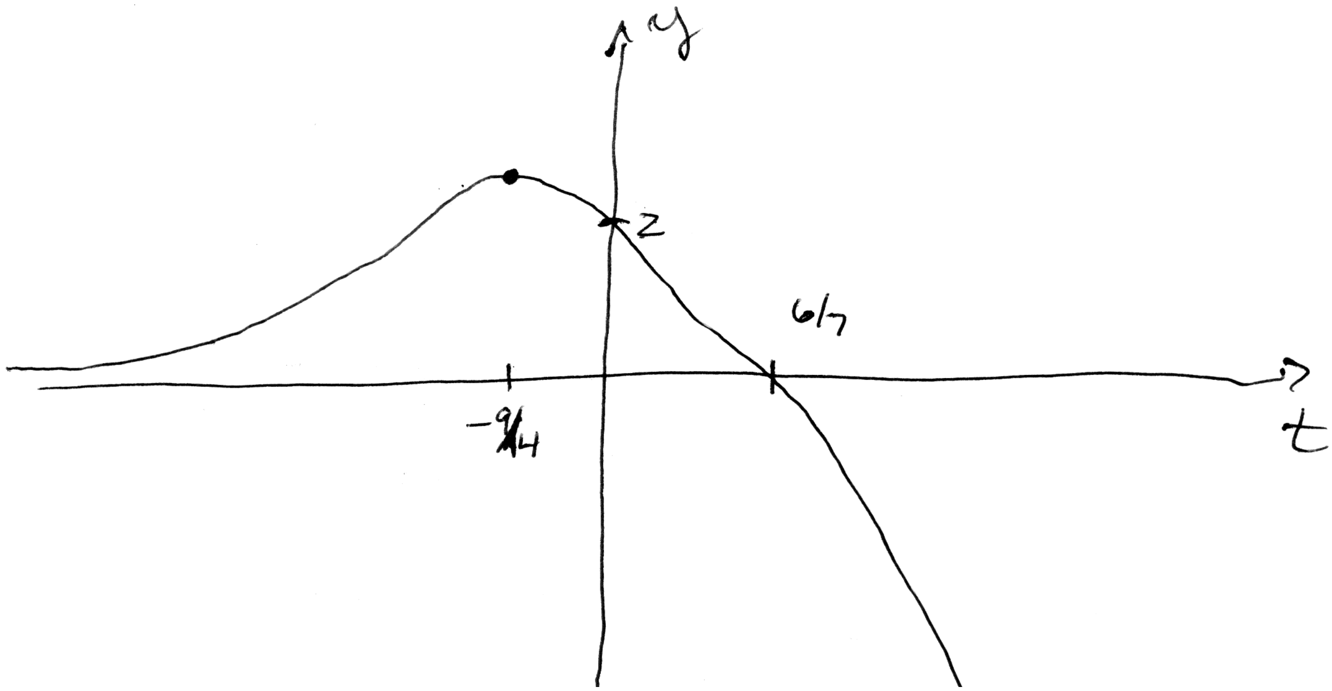
$$-1 = \tfrac{2}{3} C + D \cdot 1$$

$$C = 2 \qquad D = -7/3$$

so $$y = 2e^{2/3 t} - 7/3\, t\, e^{2/3 t}$$

$$\left(y = (2 - 7/3\, t)\, e^{2/3 t} \right)$$

$$\lim_{t \to \infty} y = -\infty \qquad \left(2 - 7/3\, t \to -\infty \;\}\; e^{2/3 t} \to +\infty \right)$$



(12) $y'' - 6y' + 9y = 0 \qquad y(0) = 0 \quad y'(0) = 2$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$y = Ce^{3t} + Dte^{3t}$

$y' = 3Ce^{3t} + D(e^{3t} + 3te^{3t})$

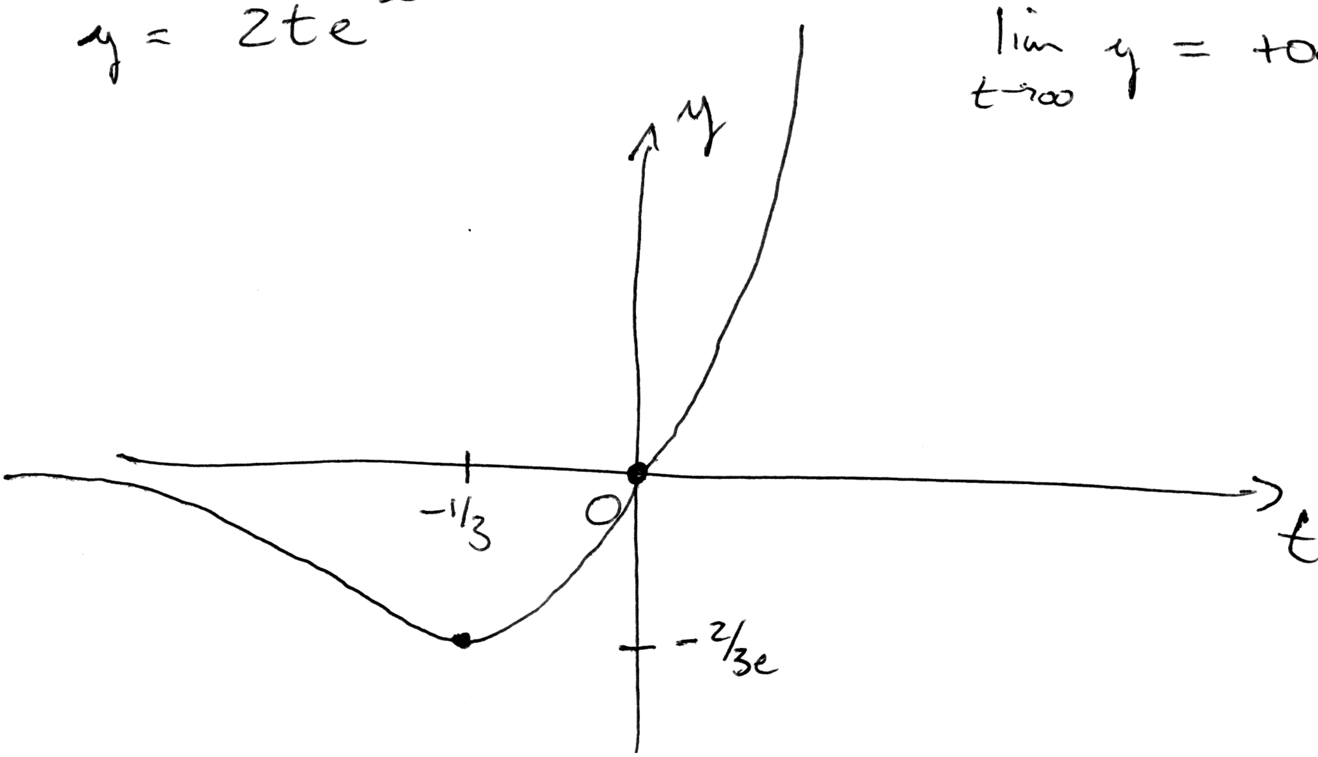
$0 = C \cdot 1 + D \cdot 0$

$2 = 3C \cdot 1 + D(1+0)$

$C = 0 \qquad D = 2$

$y = 2te^{3t}$

$\lim_{t \to \infty} y = +\infty$



(8) (a) $9y'' + 12y' + 4y = 0 \qquad y(0) = a > 0 \qquad y'(0) = -1$

$$9r^2 + 12r + 4 = 0$$

$$(3r + 2)^2 = 0$$

$$y = Ce^{-2/3t} + Dte^{-2/3t}$$

$y' = -\frac{2}{3}Ce^{-2/3t} + D\left(e^{-2/3t}\right.$ [redacted] $\left.\frac{2}{3}te^{-2/3t}\right)$

$$a = C \cdot 1 + D \cdot 0$$

$$-1 = -\frac{2}{3}C + D(1 - 0)$$

$$C = a \qquad D = \frac{2}{3}a - 1$$

$$y = ae^{-2/3t} + \left(\frac{2}{3}a - 1\right)te^{-2/3t}$$

$$y = \left(a + \left(\frac{2}{3}a - 1\right)t\right)e^{-2/3t}$$

(b) There is ~~no~~ only one sol'n y that is always positive ($e^{-2/3t} > 0$ for all t, but the line $a + \left(\frac{2}{3}a - 1\right)t$ takes on both positive & negative values for any "a" except $a = 3/2$).

If you change the question to "always positive for $t \geq 0$," then $a + \left(\frac{2}{3}a - 1\right)t$ is positive for $a \geq 3/2$ (start pos. & pos. slope).

For $a < 3/2$, there will be some t where $y < 0$.

Thus $a = 3/2$ is the cut-off sought.