§3.6

⑥ $y'' + 2y' + y = 2e^{-t}$

$y_H$: $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$y_H = C_1 e^{-t} + C_2 te^{-t}$

$y_I$: $\underline{1}\left(y_I = (Ae^{-t}) \cdot t^2\right)$

$2\left(y_I' = A\left(2te^{-t} - t^2e^{-t}\right)\right)$

$+ \quad \underline{1}\left(y_I'' = A\left(-4te^{-t} + t^2e^{-t} + 2e^{-t}\right)\right)$

---

$2e^{-t} = A\left(2e^{-t}\right)$

$\Rightarrow \quad 2 = 2A$

$\underline{1} = A \longrightarrow y_I = t^2e^{-t}$

---

$y = y_H + y_I$

$y = C_1 e^{-t} + C_2te^{-t} + t^2e^{-t}$

7 $\quad 2y'' + 3y' + y = \underset{\mathrm{I}}{t^2} + \underset{\mathrm{II}}{3\sin t} + \underset{H}{0}$

$$y_H: \quad 2r^2 + 3r + 1 = 0$$

$$(2r+1)(r+1) = 0$$

$$y_H = C_1 e^{-t/2} + C_2 e^{-t}$$

---

$$1\left(y_I = At^2 + Bt + C\right)$$

$$3\left(y_I' = 2At + B\right)$$

$$+\,2\left(y_I'' = 2A\right)$$

---

$$t^2 = At^2 + (6A+B)t + (4A+3B+C)$$

$1 = A \qquad (t^2 \text{ term})$

$0 = 6A + B \qquad (t \text{ term})$

$0 = 4A + 3B + C \qquad (1 \text{ term})$

$A = 1 \qquad B = -6 \qquad C = 14$

$$y_I = t^2 - 6t + 14$$

---

⑦ (cont)

$$1\,(y_{II} = A \sin t + B \cos t)$$

$$3\,(y'_{II} = -B \sin t + A \cos t)$$

$$2\,(y''_{II} = -A \cos t \text{ [redacted] } B \cos t)$$

---

$$3 \sin t = (A - 3B - 2A) \sin t + (B + 3A - 2B) \cos t$$

$$3 = -A - 3B \quad (\sin t \text{ term})$$

$$0 = 3A - B \quad (\cos t \text{ term})$$

$$A = -3/10 \qquad B = -9/10$$

$$y_{II} = -\frac{3}{10} \sin t - \frac{9}{10} \cos t$$

---

$$y = y_H + y_I + y_{II}$$

$$y = C_1 e^{-t/2} + C_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10} \sin t - \frac{9}{10} \cos t$$

---

⑧ $y'' + y = 3 \sin 2t + t \cos 2t$

$y_H$: $r^2 + 1 = 0$

$$r = \pm i$$

$$y_H = C_1 \sin t + C_2 \cos t$$

8 (cont) $y_I = (At+B)\sin 2t + (Ct+D)\cos 2t$

$0 \quad y_I' = [A - 2(Ct+D)]\sin 2t + [C + 2(At+B)]\cos 2t$

$1 \quad y_I'' = [-4C - 4(At+B)]\sin 2t + [4A - 4(Ct+D)]\cos 2t$

---

$$\begin{aligned} 3\sin 2t + t\cos 2t &= [-4C - 3(At+B)]\sin 2t \\ &\quad + [4A - 3(Ct+D)]\cos 2t \end{aligned}$$

$3 = -4C - 3B$ (sin 2t term)

$0 = -3A$ (t sin 2t term)

$0 = 4A - 3D$ (cos 2t term)

$1 = -3C$ (t cos 2t term)

$A = 0 \quad D = 0 \quad C = -1/3 \quad B = -5/9$

$y_I = (0t - 5/9)\sin 2t + (-1/3\, t + 0)\cos 2t$

$$y = C_1 \sin t + C_2 \cos t - \frac{5}{9}\sin 2t - \frac{1}{3} t \cos 2t$$

Yes, all the terms in $y_I$ are necessary for a correct computation. I've written slightly simplified forms for $y_I'$ & $y_I''$ to save space.

(9) $y'' + \omega_0^2 y = \cos(\omega t)$ w/ $\omega_0^2 \neq \omega^2$

(that is, $\omega_0 \neq \omega$ and $\omega_0 \neq -\omega$)

$y_H$ : $r^2 + \omega_0^2 = 0$

$$r = \pm i\omega_0$$

$$y_H = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t)$$

$$\omega_0^2 \left( y_I = A\sin(\omega t) + B\cos(\omega t) \right)$$
$$0 \left( y_I' = (-B\omega)\sin(\omega t) + (A\omega)\cos(\omega t) \right)$$
$$1 \left( y_I'' = (-A\omega^2)\sin(\omega t) + (-B\omega^2)\cos(\omega t) \right)$$

---

$$\cos(\omega t) = \left[-A\omega^2 + A\omega_0^2\right]\sin(\omega t) + \left[-B\omega^2 + B\omega_0^2\right]\cos(\omega t)$$

$$0 = -A\omega^2 + A\omega_0^2$$
$$1 = -B\omega^2 + B\omega_0^2$$

$(\omega_0^2 - \omega^2 \neq 0)$

$$A = 0 \qquad B = 1/(\omega_0^2 - \omega^2)$$

$$y = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t) + \left(\frac{1}{\omega_0^2 - \omega^2}\right)\cos(\omega t)$$

(13) $y'' + y' - 2y = 2t \quad y(0) = 0 \quad y'(0) = 1$

$y_H$: $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$y_H = C_1 e^{-2t} + C_2 e^t$

$-2\,(y_I = At + B)$

$1\,(y_I' = A)$

$1\,(y_I'' = 0)$

---

$2t = -2At + (A - 2B)$

$0 = A - 2B$

$2 = -2A$

$A = -1 \qquad B = -1/2$

$y_I = -t - 1/2$

$y = y_H + y_I$

$y = C_1 e^{-2t} + C_2 e^t - t - 1/2$

$y' = -2C_1 e^{-2t} + C_2 e^t - 1$

13 (cont)

$$0 = C_1 + C_2 - 1/2$$

$$1 = -2C_1 + C_2 - 1$$

$$C_1 = -1/2 \qquad C_2 = 1$$

$$y = -\frac{1}{2}e^{-2t} + e^{t} - t - 1/2$$

12 $y'' + 4y = 3\sin(2t) \qquad y(0) = 2 \quad y'(0) = -1$

$$y_H : r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y_H = C_1 \sin(2t) + C_2 \cos(2t)$$

$$4\,(\; y_I = \left(A \sin(2t) \cancel{t} + B \cos(2t)\right) t \;)$$

$$0\,(\; y_I' = (-2Bt + A)\sin(2t) + (2At + B)\cos(2t) \;)$$

$$1\,(\; y_I'' = (-4At - 4B)\sin(2t) + (-4Bt + 4A)\cos(2t) \;)$$

---

$$3\sin 2t = (-4B)\sin(2t) + (4A)\cos(2t)$$

$$3 = -4B \qquad A = 0$$

$$0 = 4A \qquad B = -3/4$$

$$y_I = -3/4\, t \cos(2t)$$

(17) (cont)

$$y = C_1 \sin(2t) + C_2 \cos(2t) - \frac{3}{4}t\cos(2t)$$

$$y' = 2C_1\cos(2t) - 2C_2\sin(2t) - \frac{3}{4}\cos(2t) + \frac{3}{4}t \cdot 2\sin(2t)$$

$$2 = C_1 \cdot 0 + C_2 \cdot 1 - 0$$

$$-1 = 2C_1 \cdot 1 - 2C_2 \cdot 0 - \frac{3}{4} + 0$$

$$C_1 = -1/8 \qquad C_2 = 2$$

$$y = -\frac{1}{8}\sin(2t) + 2\cos(2t) - \frac{3}{4}t\cos(2t)$$

§3.7

(2) $y'' - y' - 2y = 2e^{-t}$

$y_H$: $r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$

$$y_H = C_1 e^{2t} + C_2 e^{-t}$$

$$y_I = u_1 e^{2t} + u_2 e^{-t}$$

---

$$0 = u_1' e^{2t} + u_2' e^{-t}$$

$$2e^{-t} = u_1' 2e^{2t} + u_2'(-e^{-t})$$

$$u_1' = -\frac{e^{-t} \cdot 2e^{-t}}{W(e^{2t}, e^{-t})} \qquad u_2' = \frac{e^{2t} \cdot 2e^{-t}}{W(e^{2t}, e^{-t})}$$