§5.2

(2) $y'' - xy' - y = 0$

$$-1\left(y = \sum_0^\infty a_n x^n\right)$$

$$-x\left(y' = \sum_1^\infty n a_n x^{n-1}\right)$$

$$+1\left(y'' = \sum_2^\infty n(n-1) a_n x^{n-2}\right)$$

$$0 = \sum_2^\infty n(n-1) a_n x^{n-2} + (-x)\sum_1^\infty n a_n x^{n-1} + (-1)\sum_0^\infty a_n x^n$$

$$0 = \sum_2^\infty n(n-1) a_n x^{n-2} + \sum_1^\infty \left(-n a_n x^n\right) + \sum_0^\infty \left(-a_n x^n\right)$$

$$0 = \sum_0^\infty (n+2)(n+1) a_{n+2} x^n + \sum_1^\infty \left(-n a_n x^n\right) + \sum_0^\infty \left(-a_n x^n\right)$$

② (cont)

$$0 = 2a_2 + \sum_1^\infty (n+2)(n+1)a_{n+2}x^n$$
$$+ \sum_1^\infty (-na_n x^n)$$
$$+ (-a_0) + \sum_1^\infty (-a_n x^n)$$

$$0 = (2a_2 - a_0) + \sum_1^\infty (n+2)(n+1)a_{n+2}x^n - na_n x^n - a_n x^n$$

$$0 = (2a_2 - a_0) + \sum_1^\infty \left[(n+2)(n+1)a_{n+2} - (n+1)a_n\right] x^n$$

($0 =$ some polynomial (for every x))

every coeff in that polynomial is 0

so that

$$0 = 2a_2 - a_0 \quad \text{(coeff of const term)}$$
$$0 = (n+2)(n+1)a_{n+2} - (n+1)a_n \quad \text{(coeff of linear, quadratic, cubic, et cetera terms; } n \geq 1)$$

Writing these in a recursive way:

$$a_2 = \frac{1}{2}a_0$$
$$a_{n+2} = \frac{a_n}{n+2} \quad (n \geq 1)$$

These together are the recursion relations for the a_n's.

② (cont)

Going back to our posed sol'n:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

And applying what we know from the rec. rel'ns:

$$a_2 = \tfrac{1}{2} a_0$$

$$a_3 = \tfrac{1}{3} a_1$$

$$a_4 = \tfrac{1}{4} a_2 = \tfrac{1}{8} a_0$$

$$a_5 = \tfrac{1}{5} a_3 = \tfrac{1}{15} a_1$$

We get:

$$y = a_0 + a_1 x + \tfrac{1}{2} a_0 x^2 + \tfrac{1}{3} a_1 x^3 + \tfrac{1}{8} a_0 x^4 + \tfrac{1}{15} a_1 x^5 + \dots$$

Segregating terms into those w/ a_0 & those w/ a_1:

$$y = a_0 \left(1 + 0x + \tfrac{1}{2} x^2 + 0x^3 + \tfrac{1}{8} x^4 + 0x^5 + \dots\right)$$
$$+ a_1 \left(0 + 1x + 0x^2 + \tfrac{1}{3} x^3 + 0x^4 + \tfrac{1}{15} x^5 + \dots\right)$$

$$y = a_0 y_1 + a_1 y_2$$

where

$$y_1 = 1 + \tfrac{1}{2} x^2 + \tfrac{1}{8} x^4 + \dots \quad \& \quad y_2 = x + \tfrac{1}{3} x^3 + \tfrac{1}{15} x^5 + \dots$$

are two linearly independent sol'ns to the DE.

(5) $(1-x)y'' + y = 0$

$$
\begin{aligned}
1 \quad & \left(y = \sum_{0}^{\infty} a_n x^n \right) \\
0 \quad & \left(y' = \sum_{1}^{\infty} n a_n x^{n-1} \right) \\
+ (1-x) \quad & \left(y'' = \sum_{2}^{\infty} n(n-1) a_n x^{n-2} \right)
\end{aligned}
$$

$$0 = \sum_{0}^{\infty} a_n x^n + (1-x) \sum_{2} n(n-1) a_n x^{n-2}$$

$$0 = \sum_{0}^{\infty} a_n x^n + \sum_{2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{2}^{\infty} -n(n-1) a_n x^{n-1}$$

$$0 = \sum_{0}^{\infty} a_n x^n + \sum_{0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{1}^{\infty} -(n+1)(n) a_{n+1} x^n$$

$$0 = a_0 + 2a_2 + \sum_{1}^{\infty} \left[a_n + (n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} \right] x^n$$

$$0 = a_0 + 2a_2$$

$$0 = a_n + (n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} \qquad (\text{for } n \geq 1)$$

Recursion relations are:

$$
\begin{cases}
a_2 = -\frac{1}{2} a_0 \\
a_{n+2} = \dfrac{n(n+1) a_{n+1} - a_n}{(n+2)(n+1)} & (n \geq 1)
\end{cases}
$$

(5) (cont)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$a_2 = -\frac{1}{2} a_0$$

$$a_3 = \frac{2a_2 - a_1}{6} = -\frac{1}{6} a_0 - \frac{1}{6} a_1$$

$$a_4 = \frac{6a_3 - a_2}{12} = \frac{1}{2}\left(-\frac{1}{6} a_0 - \frac{1}{6} a_1\right) - \frac{1}{12}\left(-\frac{1}{2} a_0\right) = -\frac{1}{24} a_0 - \frac{1}{12} a_1$$

$$a_5 = \frac{12a_4 - a_3}{20} = \frac{1}{20}\left[-\frac{1}{2} a_0 - a_1 + \frac{1}{6} a_0 + \frac{1}{6} a_1\right] = -\frac{1}{60} a_0 - \frac{1}{24} a_1$$

$$y = a_0 + a_1 x + \left(-\frac{1}{2} a_0\right) x^2 + \left(-\frac{1}{6} a_0 - \frac{1}{6} a_1\right) x^3 + \left(-\frac{1}{24} a_0 - \frac{1}{12} a_1\right) x^4 + \left(-\frac{1}{60} a_0 - \frac{1}{24} a_1\right) x^5 + \cdots$$

$$= a_0 \left(1 + 0x - \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{24} x^4 - \frac{1}{60} x^5 + \cdots\right)$$

$$+ a_1 \left(0 + 1x + 0x^2 - \frac{1}{6} x^3 - \frac{1}{12} x^4 - \frac{1}{24} x^5 + \cdots\right)$$

Two indep sol'ns are:

$$y_1 = 1 - \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{24} x^4 - \frac{1}{60} x^5 + \cdots$$

$$y_2 = x - \frac{1}{6} x^3 - \frac{1}{12} x^4 - \frac{1}{24} x^5 + \cdots$$

⑥ $(2+x^2)y'' - xy' + 4y = 0$

$$4\left(y = \sum_0^\infty a_n x^n\right)$$

$$-x\left(y' = \sum_1^\infty n a_n x^{n-1}\right)$$

$$+\ (2+x^2)\left(y'' = \sum_2^\infty n(n-1) a_n x^{n-2}\right)$$

$$0 = \sum_0^\infty 4a_n x^n + \sum_1^\infty -n a_n x^n$$

$$+ \underbrace{\sum_2^\infty 2n(n-1) a_n x^{n-2}}_{\sum_0^\infty 2(n+2)(n+1) a_{n+2} x^n} + \sum_2^\infty n(n-1) a_n x^n$$

$$0 = 4a_0 + 4a_1 x - a_1 x + 4a_2 + 12 a_3 x + \sum_2^\infty \left[4a_n - n a_n + 2(n+2)(n+1) a_{n+2} + n(n-1)a_n\right] x^n$$

$$0 = 4a_0 + 4a_2$$

$$0 = 4a_1 - a_1 + 12 a_3$$

$$0 = 4a_n - n a_n + 2(n+2)(n+1) a_{n+2} + n(n-1) a_n \qquad (n \geq 2)$$

$$\begin{cases} a_2 = -a_0 \\ a_3 = -\frac{1}{4} a_1 \\ a_{n+2} = \dfrac{(-4 + n - n(n-1))\, a_n}{2(n+2)(n+1)} \qquad (n \geq 2) \end{cases}$$

$-n^2 + 2n - 4$

⑥ (cont)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$a_2 = -a_0$$

$$a_3 = -\frac{1}{4} a_1$$

$$a_4 = \frac{(-4)a_2}{24} = \frac{1}{6} a_0$$

$$a_5 = \frac{(-7)a_3}{40} = \frac{7}{160} a_1$$

$$y = a_0 + a_1 x + (-a_0)x^2 + \left(-\frac{1}{4}a_1\right)x^3 + \frac{1}{6}a_0 x^4 + \frac{7}{160} x^5 + \cdots$$

$$y = a_0\left(1 + 0x - 1x^2 + 0x^3 + \frac{1}{6}x^4 + 0x^5 + \cdots\right)$$
$$+ a_1\left(0 + 1x + 0x^2 - \frac{1}{4}x^3 + 0x^4 + \frac{7}{160}x^5 + \cdots\right)$$

Two indep sol'ns are

$$y_1 = 1 - x^2 + \frac{1}{6}x^4 + \cdots$$

$$y_2 = x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \cdots$$

In what follows:

$$y = a_0 + \sum_1^\infty a_n x^n \longrightarrow y(0) = a_0$$

$$y' = a_1 + \sum_2^\infty n a_n x^{n-1} \longrightarrow y'(0) = a_1$$

15 $\underbrace{y'' - xy' - y = 0}$ $y(0)=2$ $y'(0)=1$

#2

$$y = a_0\left(1 + \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 + \cdots\right) + a_1\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{15}x^5 + \cdots\right)$$

So

$$y = 2 + x + x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{15}x^5 + \cdots$$

since $a_0 = 2$ & $a_1 = 1$ (from the ICs)

16 $\underbrace{(2+x^2)y'' - xy' + 4y = 0}$ $y(0)=-1$ $y'(0)=3$

#6

$$y = a_0\left(1 - x^2 + \tfrac{1}{6}x^4 + \cdots\right) + a_1\left(x - \tfrac{1}{4}x^3 + \tfrac{7}{160}x^5 + \cdots\right)$$

ICs $\longrightarrow$ $a_0 = -1$ & $a_1 = 3$ so that

$$y = -1 + 3x + x^2 - \tfrac{3}{4}x^3 - \tfrac{1}{6}x^4 + \tfrac{21}{160}x^5 + \cdots$$