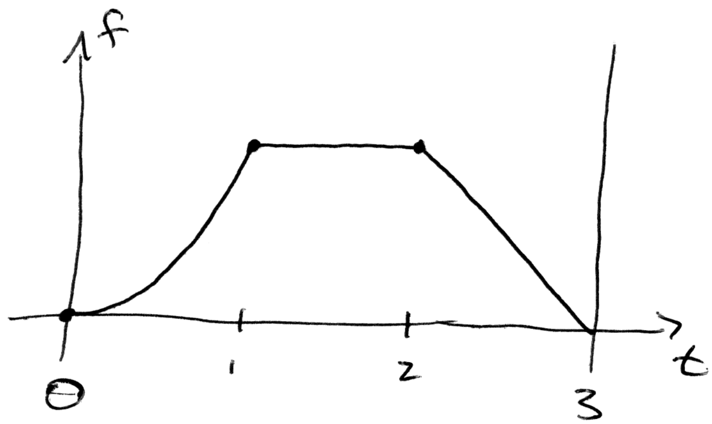
§6.2

(3) $$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \\ 3-t & 2 < t \le 3 \end{cases}$$



f is continuous everywhere in [0,3]

(5)(a) $$\mathcal{L}(t)(s) = \int_0^\infty e^{-ts}\, t\, dt$$

$$(s \neq 0) \qquad = (t)\left(-\frac{e^{-ts}}{s}\right)\Bigg]_0^\infty - \int_0^\infty \left(-\frac{e^{-ts}}{s}\right) dt$$

$$= -\frac{te^{-ts}}{s} - \frac{e^{-ts}}{s^2}\Bigg]_0^\infty$$

$$(s > 0) \qquad = \left(-\frac{0}{s} - \frac{0}{s^2}\right) - \left(-\frac{0}{s} - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

5 b

$$\mathcal{L}(t^2)(s) = \int_0^\infty e^{-ts}\, t^2\, dt$$

$$(s \neq 0) \qquad = (t^2)\left(-\frac{e^{-ts}}{s}\right)\Big]_0^\infty - \int_0^\infty \left(-\frac{e^{-ts}}{s}\right)(2t\,dt)$$

$$(s > 0) \qquad = 0 + \frac{2}{s}\int_0^\infty e^{-ts} \cdot t\,dt$$

$$= \frac{2}{s}\cdot\left(\mathcal{L}(t)(s)\right)$$

$$= \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

5 c $\mathcal{L}(t^n)(s) = \dfrac{n!}{s^{n+1}}$ for n a pos. integer

Pf: Use induction.

Base case: $n=1$ (part a)

$$\mathcal{L}(t)(s) = \frac{1!}{s^{1+1}} = \frac{1}{s^2} \quad \checkmark$$

Inductive case: pick any n & use part b-like techniques

$$\mathcal{L}(t^n)(s) = \int_0^\infty e^{-st}\, t^n\, dt$$

$$= t^n\left(-\frac{e^{-st}}{s}\right)\Big]_0^\infty - \int_0^\infty \left(-\frac{e^{-ts}}{s}\right) n t^{n-1}\, dt$$

$$= \frac{n}{s}\int_0^\infty e^{-ts} t^{n-1}\, dt = \frac{n}{s}\cdot\frac{(n-1)!}{s^{n-1+1}} = \frac{n!}{s^{n+1}} \quad \checkmark$$

(7) We could use direct computation from the definition to get $\mathcal{L}(\cosh(bt))(s)$, but I'm going to use Example #5 & the linearity of the Laplace transform (Eqn #5)

$$\mathcal{L}(\cosh(bt))(s) = \mathcal{L}\left(\frac{1}{2}\left(e^{bt}+e^{-bt}\right)\right)(s) \quad \left(\begin{array}{c}\text{def'n of}\\ \cosh\end{array}\right)$$

$$= \left(\frac{1}{2}\mathcal{L}(e^{bt}) + \frac{1}{2}\mathcal{L}(e^{-bt})\right)(s) \quad (\text{linearity})$$

$$(s>|b|) \qquad = \frac{1}{2}\cdot\frac{1}{s-b} + \frac{1}{2}\cdot\frac{1}{s+b} \quad (\text{Ex \#5})$$

$$= \frac{s}{s^2-b^2} \quad (\text{algebra})$$

(9) $$e^{at}\cosh(bt) = e^{at}\cdot\frac{1}{2}\left(e^{bt}+e^{-bt}\right)$$

$$= \frac{1}{2}e^{(a+b)t} + \frac{1}{2}e^{(a-b)t}$$

Thus (for $s > a+|b|$)

$$\mathcal{L}(e^{at}\cosh(bt))(s) = \frac{1}{2}\,\frac{1}{s-(a+b)} + \frac{1}{2}\,\frac{1}{s-(a-b)}$$

$$= \frac{s-a}{(s-a)^2-b^2}$$

(13) $e^{at}\sin(bt) = e^{at}\cdot\frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)$

$$= \frac{1}{2i}e^{(a+ib)t} - \frac{1}{2i}e^{(a-ib)t}$$

Thus (for $s>a$)

$$\mathcal{L}\left(e^{at}\sin(bt)\right)(s) = \frac{1}{2i}\cdot\frac{1}{s-(a+bi)} - \frac{1}{2i}\,\frac{1}{s-(a-ib)}$$

$$= \frac{b}{(s-a)^2+b^2}$$