§6.2

③ $$\frac{2}{s^2+3s-4} = \frac{2}{(s+4)(s-1)}$$

$$= \frac{A}{s+4} + \frac{B}{s-1}$$

$A = -2/5$ (mult. by $s+4$ & eval @ $s=-4$)

$B = 2/5$ (mult. by $s-1$ & eval @ $s=1$)

$$= -\frac{2}{5}\cdot\frac{1}{s+4} + \frac{2}{5}\,\frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^2+3s-4}\right) = -\frac{2}{5}\,\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) + \frac{2}{5}\,\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= -\frac{2}{5}e^{-4t} + \frac{2}{5}e^{t}$$ (use rule #2 p304)

⑬ $y'' - 2y' + 2y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$

$$\mathcal{L}(y'' - 2y' + 2y) = \mathcal{L}(0)$$

$$\|$$

$$(s^2 Y - s\cdot 0 - 1) - 2(sY - 0) + 2Y \qquad \text{if } \mathcal{L}(y) = Y$$

so that

$$(s^2 - 2s + 2)Y - 1 = 0$$

$$Y = \frac{1}{s^2-2s+2}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^2-2s+2}\right)$$

13 (cont) s^2-2s+2 has no real roots. In the table on p304, the only fcns w/ transforms that have denominators w/complex roots are sines & cosines (and sines & cosines mult. by exponentials).

Since $s^2-2s+2 = (s-1)^2+1$ by completing the square, we can use rule #9 on p304 to get:

$$y = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+1}\right) = e^t \sin t$$

15 $y''-2y'-2y=0 \qquad y(0)=2 \quad y'(0)=0$

$$\left(s^2Y - s\cdot 2 - 0\right) - 2(sY-2) - 2Y = 0$$

$$(s^2-2s-2)Y - 2s + 4 = 0$$

$$Y = \frac{2s-4}{s^2-2s-2} = \frac{A}{s-(1+\sqrt{3})} + \frac{B}{s-(1-\sqrt{3})}$$

$$A = \frac{2(1+\sqrt{3})-4}{(1+\sqrt{3})-(1-\sqrt{3})} = \frac{-2+2\sqrt{3}}{2\sqrt{3}} = 1-\tfrac{1}{3}\sqrt{3}$$

$$B = \frac{2(1-\sqrt{3})-4}{(1-\sqrt{3})-(1+\sqrt{3})} = \frac{-2-2\sqrt{3}}{-2\sqrt{3}} = 1+\tfrac{1}{3}\sqrt{3}$$

$$y = \left(1-\tfrac{1}{3}\sqrt{3}\right)e^{(1+\sqrt{3})t} + \left(1+\tfrac{1}{3}\sqrt{3}\right)e^{(1-\sqrt{3})t} \qquad \text{(rule \#2)}$$

ALT: $s^2-2s-2 = (s-1)^2 - 3$

$= (s-1) - (\sqrt{3})^2$

$$\frac{2s-4}{s^2-2s-2} = \frac{2s-4}{(s-1)^2-(\sqrt{3})^2} = \frac{2(s-1)-2}{(s-1)^2-(\sqrt{3})^2}$$

(15) (ALT cont):

$$\frac{2s-4}{s^2-2s-2} = 2\cdot\frac{s-1}{(s-1)^2-(\sqrt{3})^2} - \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{(s-1)^2-(\sqrt{3})^2}$$

$$y = 2e^t\cosh(\sqrt{3}t) - \frac{2}{\sqrt{3}}e^t\sinh(\sqrt{3}t)$$

(Use rules #7, 8, & 14; think along the lines that rules #5, 6 + rule #14 = rules #9, 10)

(18) $y^{(4)} - y = 0 \qquad y(0)=1 \quad y'(0)=0 \quad y''(0)=1 \quad y'''(0)=0$

$$(s^4Y - s^3\cdot 1 - s^2\cdot 0 - s\cdot 1 - 0) - Y = 0$$

$$(s^4-1)Y - s^3 - s = 0$$

$$Y = \frac{s^3-s}{s^4-1} = \frac{s(s^2-1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2+1}$$

$$y = \cos t \qquad \text{(rule \# 6)}$$

(23) $y'' + 2y' + y = 4e^{-t} \qquad y(0)=2 \quad y'(0)=-1$

$$(s^2Y - s\cdot 2 - (-1)) + 2(sY-2) + Y = 4\cdot\frac{1}{s+1}$$

$$(s^2+2s+1)Y - 2s - 3 = \frac{4}{s+1}$$

$$s^2+2s+1 = (s+1)^2$$

$$Y = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2} = \frac{4}{(s+1)^3} + \frac{A}{(s+1)^2} + \frac{B}{s+1}$$

$A = 1$ (mult by $(s+1)^2$ & eval @ $s=-1$) $\qquad 3/1 = A + B$ (eval @ $s=0$)

23 (cont)

$$Y = \frac{4}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{2}{s+1}$$

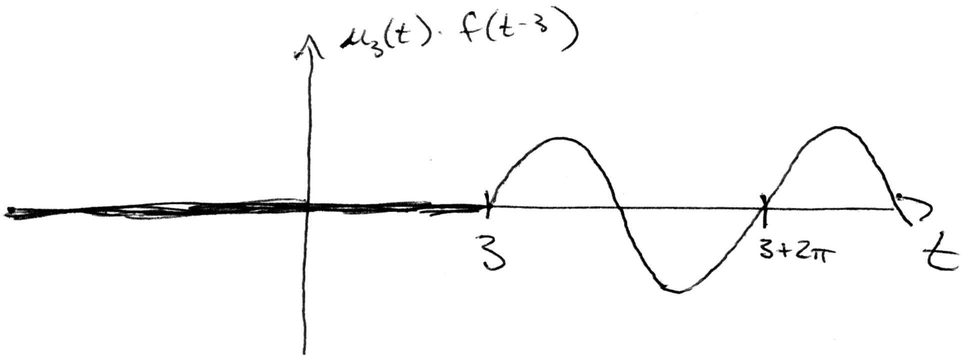
$$y = 2e^{-t} \cdot t^2 + e^{-t} \cdot t + 2e^{-t}$$

§6.3

4 $f(t) = \sin t$

$f(t-3) = \sin(t-3)$ (sin t shifted 3 units to the right)

$u_3(t)$ = unit step fcn that turns "on" @ $t=3$



7 $f(t) = \begin{Bmatrix} 0 & t<2 \\ (t-2)^2 & t \geq 2 \end{Bmatrix} = u_2(t) \cdot (t-2)^2$

$$\mathcal{L}(f(t))(s) = e^{-2s} \cdot \underbrace{\frac{2}{s^3}}$$

shift amt

$\mathcal{L}(t^2)(s)$

18 $\mathcal{L}^{-1}\left(\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$

§6.4

① $y'' + y = f \qquad y(0) = 0, \quad y'(0) = 1 \qquad f(t) = 1 - u_{\pi/2}(t)$

$$(s^2 Y - s\cdot 0 - 1) + Y = \frac{1}{s} - \frac{e^{-\pi/2\, s}}{s}$$

$$(s^2+1)Y - 1 = \frac{1 - e^{-\pi/2\, s}}{s}$$

$$Y = \frac{1 - e^{-\pi/2\, s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1} \qquad \begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned}$$

$$Y = (1 - e^{-\pi/2\, s})\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$$

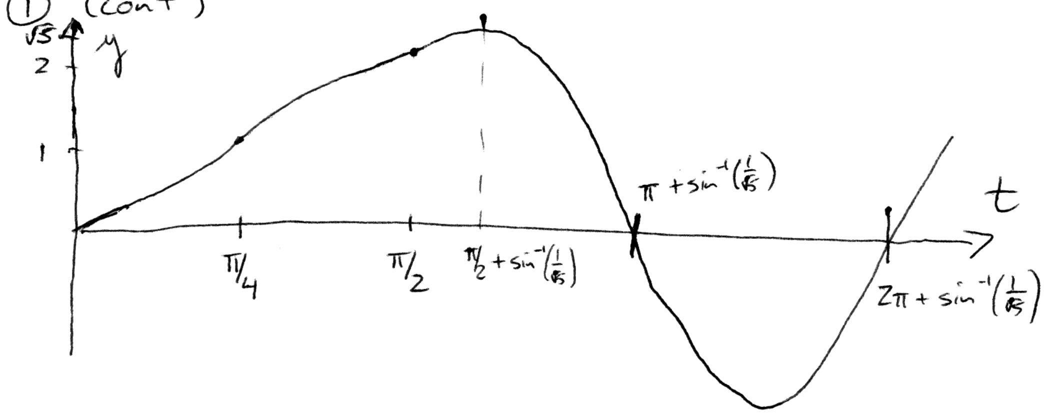
$$Y = \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{e^{-\pi/2\, s}}{s} + e^{-\pi/2\, s}\cdot\frac{s}{s^2+1}$$

$$y = 1 - \cos t + \sin t - u_{\pi/2}(t) + u_{\pi/2}(t)\cdot \underbrace{\cos(t - \pi/2)}_{\sin t}$$

$$y = 1 - \cos t + \sin t + u_{\pi/2}(t)\cdot(\sin t - 1)$$

$$= \begin{cases} 1 - \cos t + \sin t & t \le \pi/2 \\ -\cos t + 2\sin t & t \ge \pi/2 \end{cases}$$

① (cont)



(NB: both y & y' are continuous; y'' is discontinuous)

② $y'' + 2y' + 2y = h$ $\quad y(0)=0 \quad y'(0)=1$

$$h(t) = \begin{cases} 0 & 0 \le t < \pi \ \& \ 2\pi \le t \\ 1 & \pi \le t < 2\pi \end{cases}$$

$$h(t) = u_\pi(t) - u_{2\pi}(t)$$

$$(s^2Y - s\cdot 0 - 1) + 2(sY - 0) + 2Y = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$(s^2+2s+2)Y - 1 = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$Y = \frac{1}{s^2+2s+2} + \left(e^{-\pi s} - e^{-2\pi s}\right)\cdot\frac{1}{s(s^2+2s+2)}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2} \qquad A = 1/2 \quad B = -1/2 \quad C = -1$$

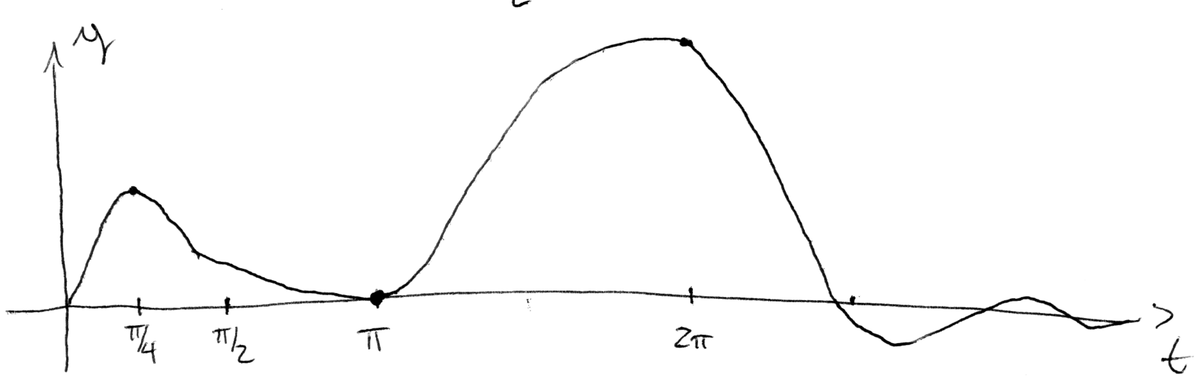
(2) (cont)

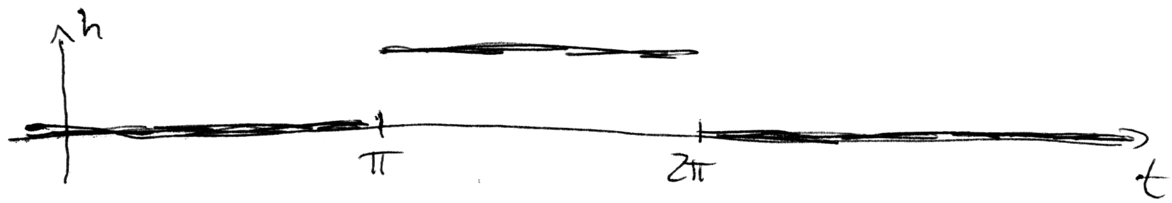
$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+2}{s^2+2s+2}$$

$$\frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1^2} + \frac{1}{(s+1)^2+1^2}$$

$$y = e^{-t}\sin t + u_{\pi}(t)\left[\frac{1}{2} - \frac{1}{2}\left(e^{-(t-\pi)}\cos(t-\pi) + e^{-(t-\pi)}\sin(t-\pi)\right)\right] - u_{2\pi}(t)\left[\frac{1}{2} - \frac{1}{2}\left(e^{-(t-2\pi)}\cos(t-2\pi) + e^{-(t-2\pi)}\sin(t-2\pi)\right)\right]$$

$$y = e^{-t}\sin t + u_{\pi}(t)\left[\frac{1}{2} + \frac{e^{\pi}}{2}\left(e^{-t}\cos t + e^{-t}\sin t\right)\right] - u_{2\pi}(t)\left[\frac{1}{2} - \frac{e^{2\pi}}{2}\left(e^{-t}\cos t + e^{-t}\sin t\right)\right]$$





④ $y'' + 4y = \sin t + u_\pi(t)\sin(t-\pi)$ $\qquad y(0)=0$, $y'(0)=0$

$$(s^2+4)Y = \frac{1}{s^2+1} + e^{-\pi s}\cdot\frac{1}{s^2+1}$$

$$Y = (1+e^{-\pi s})\cdot\frac{1}{(s^2+4)(s^2+1)}$$

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$A+C=0$

$B+D=0$

$A+4C=0$

$B+4D=1$

$A=0 \quad B=-1/3 \quad C=0 \quad D=1/3$

$$Y = (1+e^{-\pi s})\left[-\frac{1}{3}\cdot\frac{1}{s^2+4} + \frac{1}{3}\cdot\frac{1}{s^2+1}\right]$$

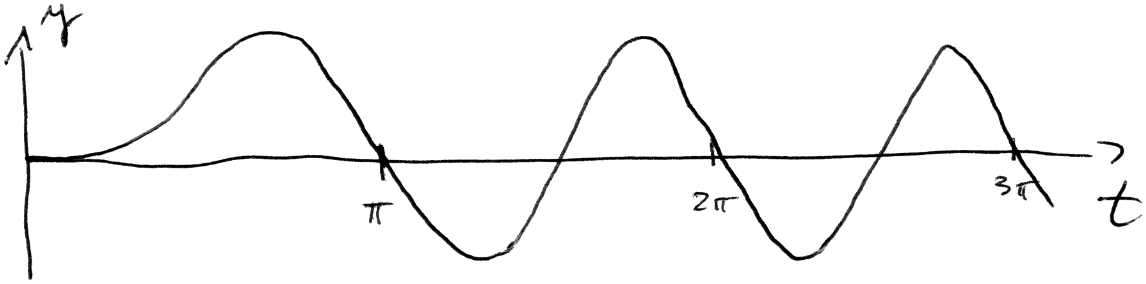
$$Y = (1+e^{-\pi s})\left[-\frac{1}{6}\cdot\frac{2}{s^2+2^2} + \frac{1}{3}\cdot\frac{1}{s^2+1^2}\right]$$

$$y = -\frac{1}{6}\sin(2t) - \frac{1}{6}u_\pi(t)\sin(2(t-\pi)) + \frac{1}{3}\sin t + \frac{1}{3}u_\pi(t)\sin(t-\pi)$$

$$y = -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t - \frac{1}{6}u_\pi(t)\sin 2t - \frac{1}{3}u_\pi(t)\sin t$$

$$\left(y = \begin{cases} \frac{1}{6}(-\sin(2t) + 2\sin t) & 0 \le t \le \pi \\ -\frac{1}{3}\sin(2t) & t \ge \pi \end{cases} \right)$$

④ (cont)



$$RHS(t) = \sin t + u_\pi(t)\sin(t-\pi) = \sin t - u_\pi(t)\sin t)$$
$$= (1-u_\pi(t))\sin t$$

⑤ $y'' + 3y' + 2y = 1 - u_{10}(t) \qquad y(0)=0 \quad y'(0)=0$

$$(s^2+3s+2)Y = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$Y = (1-e^{-10s})\frac{1}{s(s^2+3s+2)}$$

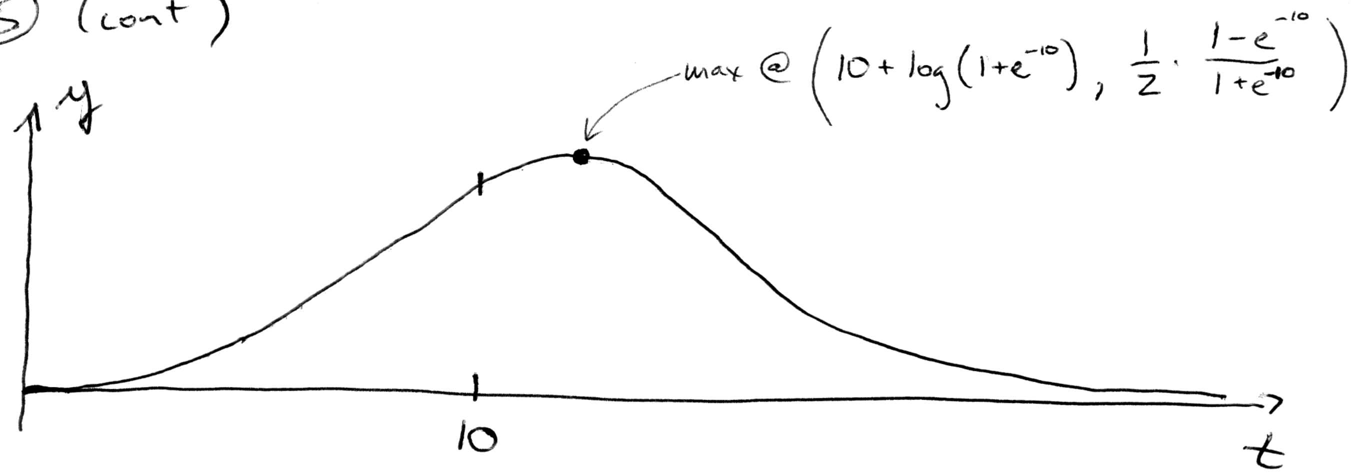
$$\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = 1/2 \quad B = -1 \quad C = 1/2$$

$$Y = (1-e^{-10s})\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}\right)$$

$$y = \left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right) - u_{10}(t)\left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right)$$

(5) (cont)



NB: The max in y comes after 10 & is less than $\frac{1}{2}$. Try verifying that with a calculator! (The scale in the graph of y above is highly nonlinear.) If it helps:

$$y = \frac{1}{2}\left(1-e^{-t}\right)^2 - u_{10}(t)\cdot\frac{1}{2}\left(1-e^{-(t-10)}\right)^2$$

(11) $y'' + 4y = u_\pi(t) - u_{3\pi}(t) \qquad y(0)=0 \quad y'(0)=0$

$$(s^2+4)Y = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}$$

$$Y = \left(e^{-\pi s} - e^{-3\pi s}\right)\cdot\frac{1}{s(s^2+4)}$$

$$Y = \left(e^{-\pi s} - e^{-3\pi s}\right)\left[\frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right]$$

(11) (cont)

$$y = \frac{1}{4} u_\pi(t)\left[1 - \cos\left(2(t-\pi)\right)\right]$$

$$+ \frac{1}{4} u_{3\pi}(t)\left[1 - \cos\left(2(t-3\pi)\right)\right]$$

Since

$$\cos\left(2(t-\pi)\right) = \cos(2t) = \cos\left(2(t-3\pi)\right)$$

Then

$$y = \frac{1}{4}\left[1 - \cos(2t)\right]\cdot\left[u_\pi(t) - u_{3\pi}(t)\right]$$