§7.2

(22) $\bar{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \bar{x}$ $\qquad \bar{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \bar{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$$

$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

$$\bar{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot 2e^{2t}$$

$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t} \quad \checkmark$$

(23) $\bar{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \bar{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$

$$\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{t}$$

$$\bar{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (e^{t} + te^{t}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{t}$$

$\overset{?}{=}$

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \bar{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{t} \right] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{t} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{t} \quad \checkmark$$

§7.5

① $\bar{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \bar{x}$

We want

Ⓐ $\det\left(\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$

and

Ⓑ $\left(\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \bar{\xi} = \bar{0}$

Ⓐ $\det \begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = 0$

$(3-r)(-2-r) - (2)(-2) = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r^{(1)} = 2$ & $r^{(2)} = -1$

Ⓑ 1st root

$\left(\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \bar{\xi}^{(1)} = \bar{0}$

$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\xi_1^{(1)} - 2\xi_2^{(1)} = 0$

$2\xi_1^{(1)} - 4\xi_2^{(1)} = 0$ (same eqn!)

① (cont)

so $\xi_1^{(1)} - 2\xi_2^{(1)} = 0$ or $\xi_1^{(1)} = 2\xi_2^{(1)}$

and $\bar{\xi}^{(1)} = \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 2\xi_2^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \xi_2^{(1)} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

still an arbitrary const

Ⓑ 2nd root

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \bar{\xi}^{(2)} = \bar{0}$$

$$2\xi_1^{(2)} - \xi_2^{(2)} = 0$$

$$\bar{\xi}^{(2)} = \xi_1^{(2)} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\bar{x} = \bar{\xi}^{(1)} e^{2t} + \bar{\xi}^{(2)} e^{-t}$$

$$= \xi_2^{(1)} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + \xi_1^{(2)} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

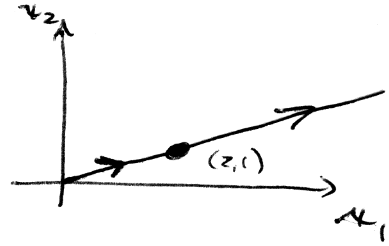
or ~~using more exact notation~~ relabeling: (to simplify the notation for the arbitrary constants)

$$\bar{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

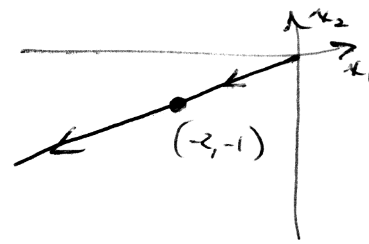
① (cont)

If we have ICs that give $C_1 = 1$ & $C_2 = 0$ then

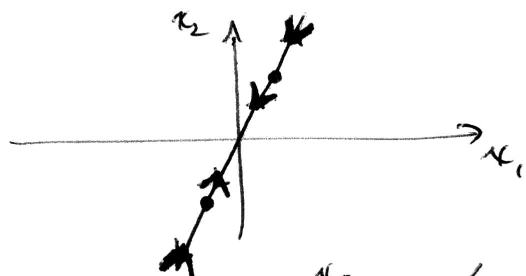
$\bar{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ with trajectory

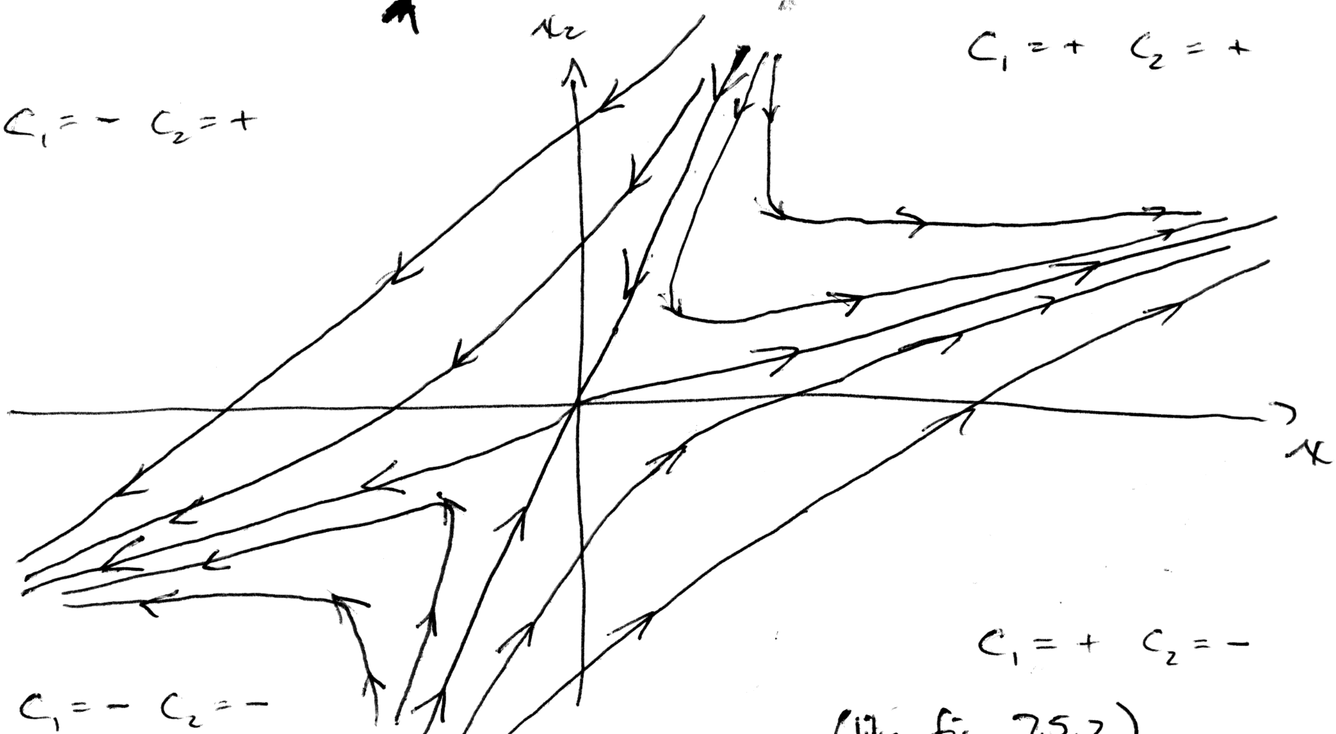


With $C_1 = -1$ & $C_2 = 0$, then $\bar{x} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{2t}$ &



With $C_1 = 0$ & $C_2 = \pm 1$ then $\bar{x} = \pm \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ and trajectories:





(like fig. 7.5.2)

① (cont)

$$\bar{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

As $t \to \infty$

$$\bar{x} \longrightarrow c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} \longrightarrow \begin{pmatrix} +\infty \\ +\infty \end{pmatrix} \cdot \operatorname{sign}(c_1)$$

② $\bar{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \bar{x}$

Ⓐ $\det \begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix} = 0$

$$(1-r)(-4-r) - (-2)(3) = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r^{(1)} = -1 \qquad r^{(2)} = -2$$

Ⓑ 1st root

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \bar{\xi}^{(1)} = \bar{0} \Rightarrow \xi_1^{(1)} - \xi_2^{(1)} = 0$$

$$\Rightarrow \bar{\xi}^{(1)} = \xi_1^{(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Ⓑ 2nd root

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \bar{\xi}^{(2)} = \bar{0} \Rightarrow 3\xi_1^{(2)} - 2\xi_2^{(2)} = 0 \Rightarrow \bar{\xi}^{(2)} = \xi_1^{(2)} \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$$

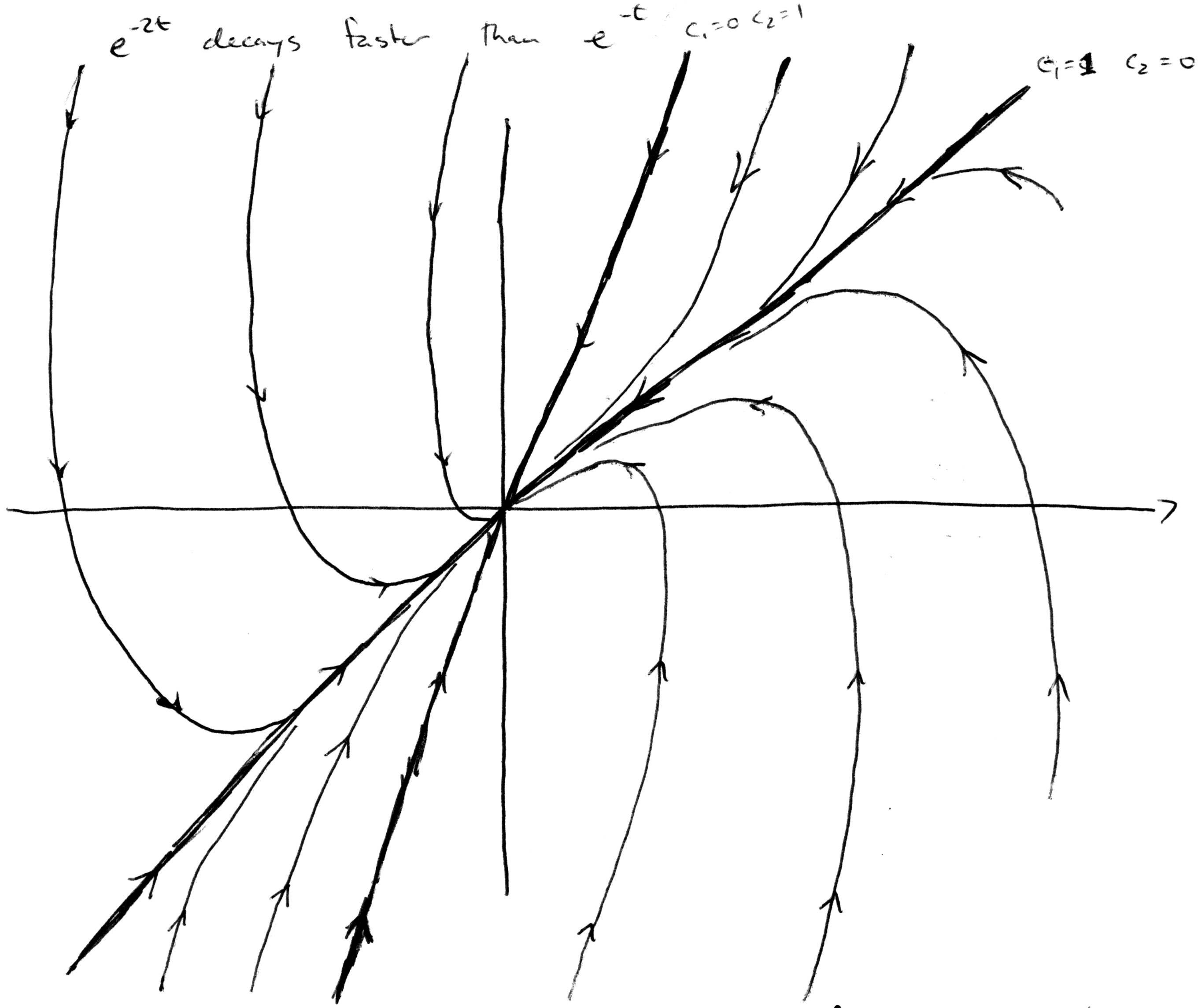
② (cont)

Thus $\bar{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} e^{-2t}$

NB: as $t \to \infty$

$$\bar{x} \longrightarrow c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{always}$$

e^{-2t} decays faster than e^{-t}



(kind of like fig. 7.5.4 but squished so e-vectors align)

④ $\bar{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \bar{x}$

Ⓐ $\det \begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} = 0$

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r^{(1)} = -3$ & $r^{(2)} = 2$

Ⓑ 1st root

$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \bar{\xi}^{(1)} = \bar{0} \Rightarrow 4\xi_1^{(1)} + \xi_2^{(1)} = 0 \Rightarrow \bar{\xi}^{(1)} = \xi_1^{(1)} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

2nd root

$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \bar{\xi}^{(2)} = \bar{0} \Rightarrow -\xi_1^{(2)} + \xi_2^{(2)} = 0 \Rightarrow \bar{\xi}^{(2)} = \xi_1^{(2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

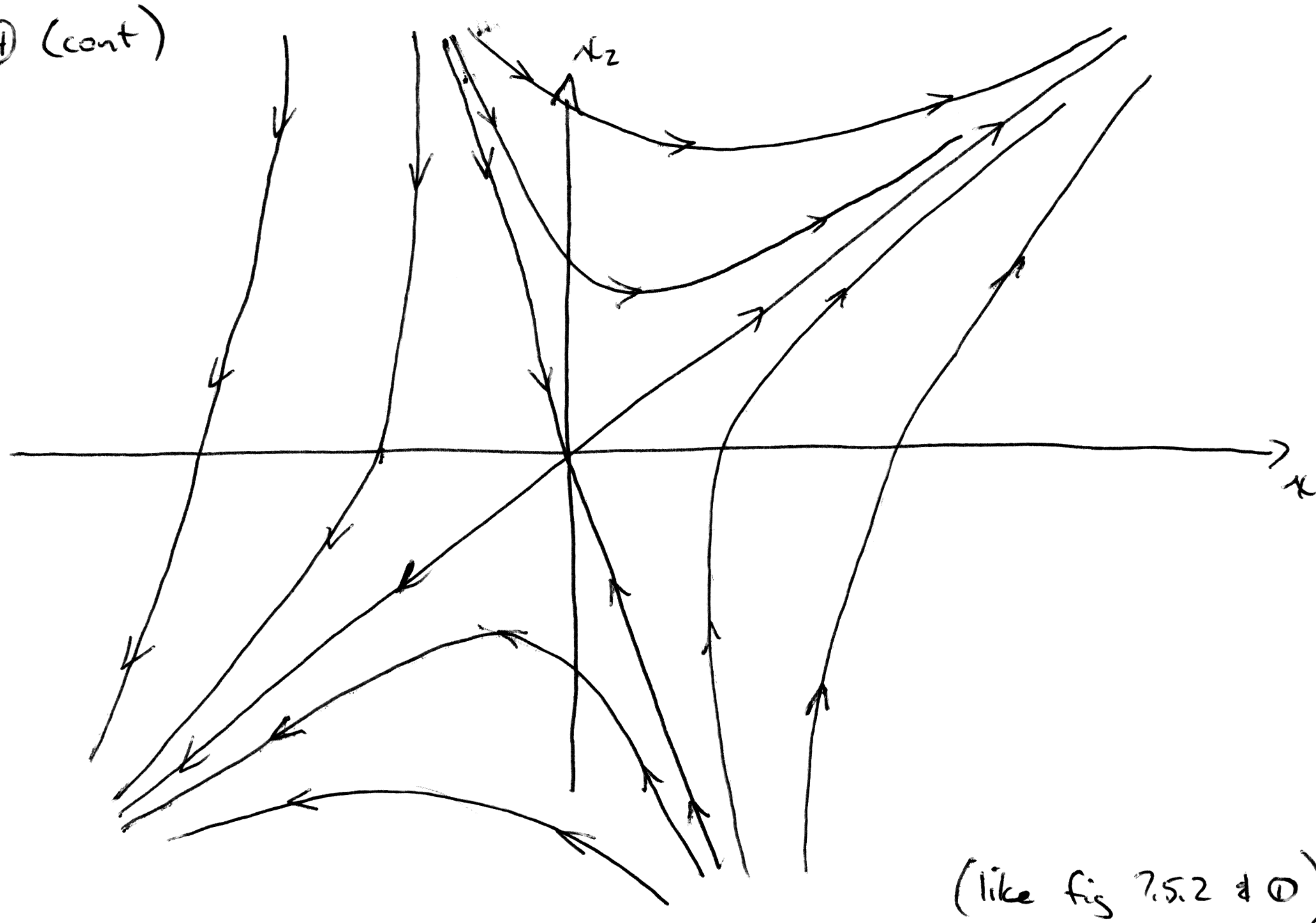
Thus

$\bar{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

As $t \to \infty$

$\bar{x} \longrightarrow c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \longrightarrow \begin{pmatrix} +\infty \\ +\infty \end{pmatrix} \cdot \operatorname{sign}(c_2)$

④ (cont)



(like fig 7.5.2 & ①)

⑧

$$\bar{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \bar{x}$$

Ⓐ $\det \begin{pmatrix} 3-r & 6 \\ -1 & -2-r \end{pmatrix} = 0$

$r^2 - r = 0$

$r^{(1)} = 0$ & $r^{(2)} = 1$

Ⓑ 1st root

$\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \xi^{(1)} = \bar{0} \Rightarrow -\xi_1^{(1)} - 2\xi_2^{(1)} = 0 \Rightarrow \xi_1^{(1)} = -2\xi_2^{(1)}$

so $\xi^{(1)} = \xi_2^{(1)} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

8 (cont)

2nd root:

$$\begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix} \bar{\xi}^{(2)} = \bar{0} \implies \xi_1^{(2)} = -3\xi_2^{(2)} \implies \bar{\xi}^{(2)} = \xi_2^{(2)} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Thus

$$\bar{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{1t}$$

or

$$\bar{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

NB: As $t \to \infty$

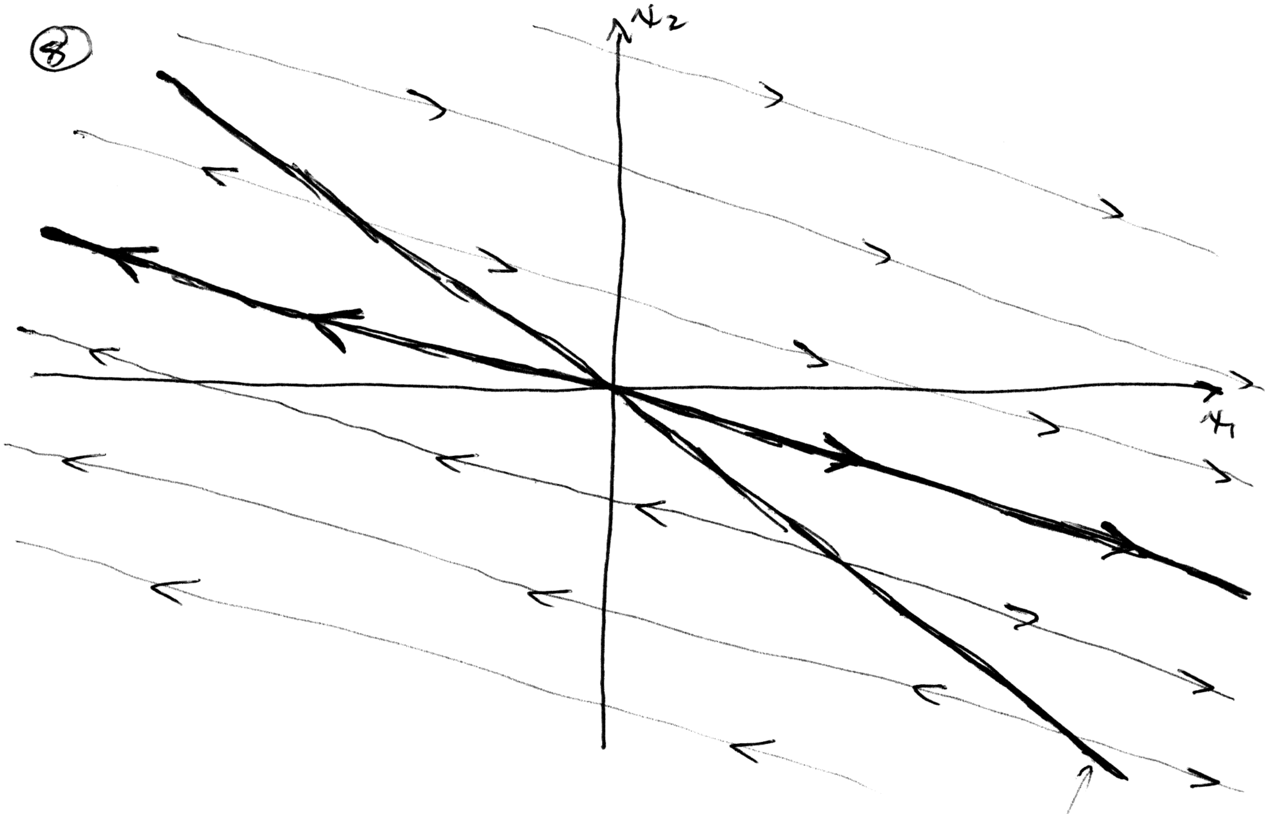
$$\bar{x} \longrightarrow c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} \longrightarrow \begin{pmatrix} -\infty \\ +\infty \end{pmatrix} \cdot \operatorname{sign}(c_2)$$

As $t \longrightarrow -\infty$

$$\bar{x} \longrightarrow c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

As t changes, only multiples of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ are added to $\bar{x}$. The component of $\bar{x}$ in the direction of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ never changes.

(8)



NB: it would be a mistake to put arrows along this line.

(15)

$$\bar{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \bar{x} \quad \& \quad \bar{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(A) $$\det \begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} = 0$$

$$r^2 - 6r + 8 = 0$$

$$r^{(1)} = 2 \quad \& \quad r^{(2)} = 4$$

(15) (cont)

1st root

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \bar{\xi}^{(1)} = \bar{0} \Rightarrow 3\xi_1^{(1)} - \xi_2^{(1)} = 0 \Rightarrow \bar{\xi}^{(1)} = \xi_1^{(1)} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2nd root

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \bar{\xi}^{(2)} = \bar{0} \Rightarrow \xi_1^{(2)} - \xi_2^{(2)} = 0 \Rightarrow \bar{\xi}^{(2)} = \xi_1^{(2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

and

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \bar{x}(0) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that

$$\begin{aligned} 2 &= c_1 + c_2 \\ -1 &= 3c_1 + c_2 \end{aligned} \Rightarrow \begin{aligned} c_1 &= -3/2 \\ c_2 &= 7/2 \end{aligned}$$

Summing up:

$$\bar{x} = (-3/2) \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + (7/2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

As $t \to \infty$, $\bar{x} \longrightarrow \begin{pmatrix} +\infty \\ +\infty \end{pmatrix}$

(NB: e^{4t} grows faster than $e^{2t} = e^{4t} \cdot e^{-2t}$)

(16) $\bar{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \bar{x}$ & $\bar{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\det \begin{pmatrix} -2-r & 1 \\ -5 & 4-r \end{pmatrix} = 0$$

$$r^2 - 2r - 3 = 0$$

$$r^{(1)} = -1 \ \& \ r^{(2)} = 3$$

$$\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \bar{\xi}^{(1)} = \bar{0} \implies \bar{\xi}^{(1)} = \xi_1^{(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \bar{\xi}^{(2)} = \bar{0} \implies \bar{\xi}^{(2)} = \xi_1^{(1)} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \bar{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\implies c_1 = 1/2 = c_2$$

$$\implies \bar{x} = (1/2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + (1/2) \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$

As $t \to \infty$, $\bar{x} \longrightarrow (1/2) \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} \longrightarrow \begin{pmatrix} +\infty \\ +\infty \end{pmatrix}$