§7.6

9. $\bar{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \bar{x} \qquad \bar{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\det \begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix} = 0$$

$$r^2 + 2r + 2 = 0$$

$r = -1 \pm i$ (say $r^{(1)} = -1+i$ & $r^{(2)} = -1-i$)

$$\begin{pmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{pmatrix} \bar{\xi}^{(1)} = \bar{0}$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

NB: $(2-i)(-2-i) = -5$

$$\xi_1^{(1)} + (-2-i)\xi_2^{(1)} = 0$$

$$\bar{\xi}^{(1)} = \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \xi_2^{(1)}$$

By conjugation:

$$\bar{\xi}^{(2)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} \xi_2^{(2)}$$

Thus

$$\bar{x} = C_1 \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \exp\left((-1+i)t\right) + C_2 \begin{pmatrix} 2-i \\ 1 \end{pmatrix} \exp\left((-1-i)t\right)$$

(9) (cont)

$$\exp((-1+i)t) = (e^{-t})(\cos t + i\sin t)$$

so that

$$c_1\begin{pmatrix}2+i\\1\end{pmatrix}\exp((-1+i)t) = c_1 e^{-t}\begin{pmatrix}2+i\\1\end{pmatrix}(\cos t + i\sin t)$$

$$= c_1 e^{-t}\begin{pmatrix}(2\cos t - \sin t) + i(\cos t + 2\sin t)\\ \cos t \quad + i \quad \sin t\end{pmatrix}$$

and (again by conjugation)

$$c_2\begin{pmatrix}2-i\\1\end{pmatrix}\exp((-1-i)t) = c_2 e^{-t}\begin{pmatrix}(2\cos t - \sin t) - i(\cos t + 2\sin t)\\ \cos t \quad - i \quad \sin t\end{pmatrix}$$

Thus

$$\bar{x} = D_1 e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} + D_2 e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\end{pmatrix}$$

$\left(\text{NB: } D_1 = \frac{c_1+c_2}{2} \;\&\; D_2 = \frac{c_1-c_2}{2i}\right)$

Using the IC & the sol'n to the DE:

$$\begin{pmatrix}1\\1\end{pmatrix} = \bar{x}(0) = D_1\begin{pmatrix}2\\1\end{pmatrix} + D_2\begin{pmatrix}1\\0\end{pmatrix}$$

Thus $D_1 = 1$ & $D_2 = -1$ and (simplifying a bit):

$$\bar{x} = e^{-t}\begin{pmatrix}\cos t - 3\sin t\\ \cos t - \sin t\end{pmatrix}$$

(10) $\bar{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \bar{x} \qquad \bar{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\det \begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix} = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = -2 \pm i$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \xi^{(1)} = \bar{0}$$

$$\xi_1^{(1)} = (1-i)\,\xi_2^{(1)} \implies \xi^{(1)} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \xi_2^{(1)}$$

by conjugation $\implies \xi^{(2)} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \xi_2^{(2)}$

$$\bar{x} = c_1 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \exp\big((-2+i)t\big) + c_2 \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \exp\big((-2-i)t\big)$$

$$\exp\big((-2+i)t\big) = e^{-2t}(\cos t + i \sin t)$$

$$c_1 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \exp\big((-2+i)t\big) = c_1 e^{-2t} \begin{pmatrix} (\cos t + \sin t) + i(-\cos t + \sin t) \\ \cos t + i \sin t \end{pmatrix}$$

$$c_2 \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \exp\big((-2-i)t\big) = c_2 e^{-2t} \begin{pmatrix} (\cos t + \sin t) - i(-\cos t + \sin t) \\ \cos t - i \sin t \end{pmatrix}$$

well, who cares...

$$\boxed{\bar{x} = D_1 e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + D_2 e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}}$$

⑩ (cont)

Using the IC & sol'n to the DE:

$$\begin{pmatrix}1\\-2\end{pmatrix} = \bar{x}(0) = D_1\begin{pmatrix}1\\1\end{pmatrix} + D_2\begin{pmatrix}-1\\0\end{pmatrix}$$

so that $D_1 = -2$ & $D_2 = -3$ and

$$\bar{x} = e^{-2t}\begin{pmatrix}\cos t - 5\sin t\\ -2\cos t - 3\sin t\end{pmatrix}$$

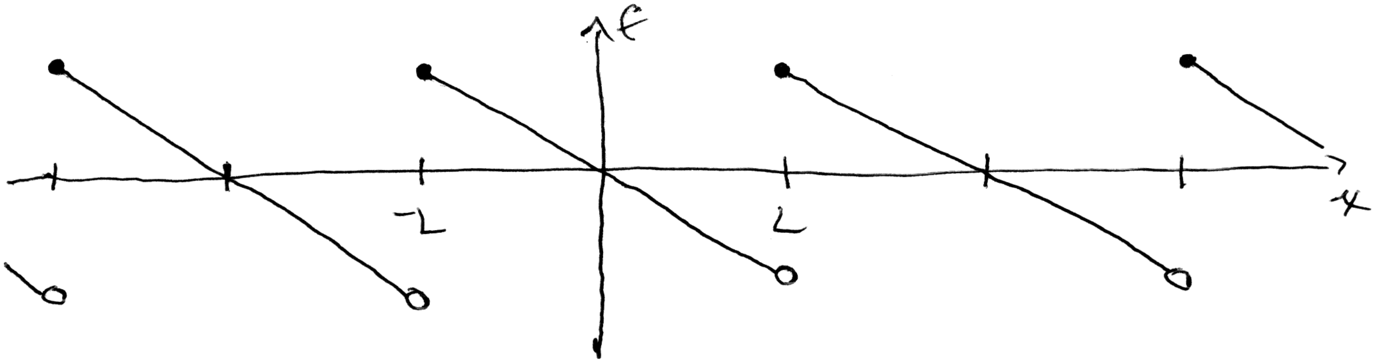
§10.2

① yes, $\sin(5x)$ is periodic w/ fundamental period $\frac{2\pi}{5}$

③ no, $\sinh(2x)$ is not periodic

⑤ yes, $\tan(\pi x)$ is periodic w/ fund. period $\frac{\pi}{\pi} = 1$

⑬ $f(x) = -x$ for $-L \le x < L$ & $f(x) = f(x+2L)$ for all x



$$f(x) = a_0/2 + \sum_1^{\infty} a_n \cos\left(n\frac{2\pi}{2L}x\right) + b_n \sin\left(n\frac{2\pi}{2L}x\right)$$

(13) (cont)

where the coefficients (a_n & b_n's) are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 0$$

$$= \frac{1}{L}\int_{-L}^{L} -x\cos\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 0$$

$$= \frac{1}{L}\left[(-x)\left(\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right) - \left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \qquad n \geq 1$$

$$= \frac{1}{L}\left[\left(-\frac{L^2}{n\pi}\sin(n\pi) - \left(\frac{L}{n\pi}\right)^2\cos(n\pi)\right) - \left(\frac{L^2}{n\pi}\sin(-n\pi) - \left(\frac{L}{n\pi}\right)^2\cos(-n\pi)\right)\right]$$

$$= 0 \qquad (n \geq 1)$$

$$= \frac{1}{L}\left[-\frac{x^2}{2}\right]_{-L}^{L} \qquad n = 0$$

$$= 0 \qquad (n = 0)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 1$$

$$= \frac{1}{L}\int_{-L}^{L} -x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[(-x)\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right) - \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$$

(13) (cont)

$$b_n = \frac{1}{L}\left[\left(\frac{L^2}{n\pi}\cos(n\pi) - \left(\frac{L}{n\pi}\right)^2 \sin(n\pi)\right) - \left(-\frac{L^2}{n\pi}\cos(-n\pi) - \left(\frac{L}{n\pi}\right)^2 \sin(-n\pi)\right)\right]$$

$\cos(-n\pi) = \cos(n\pi)$ & $\sin(-n\pi) = -\sin(n\pi)$

$$b_n = \frac{2}{L}\left[\frac{L^2}{n\pi}\cos(n\pi) - \left(\frac{L}{n\pi}\right)^2 \sin(n\pi)\right]$$

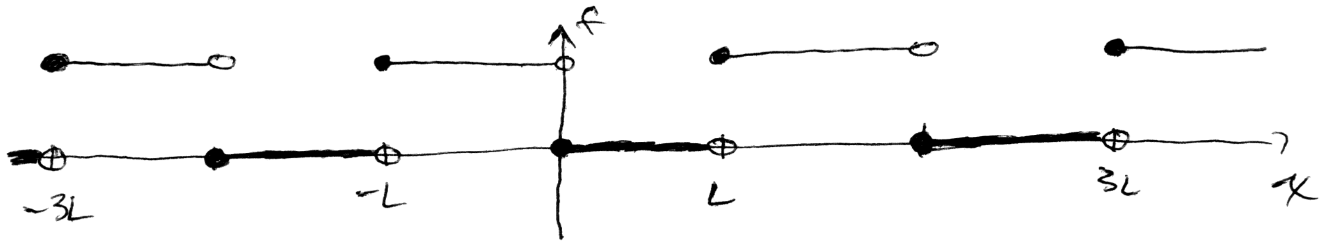
$\cos(n\pi) = (-1)^n$ & $\sin(n\pi) = 0$

$$b_n = \frac{2L}{n\pi}(-1)^n$$

Thus

$$f(x) = \sum_{1}^{\infty} \frac{2L}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{L}\right)$$

(14) $f(x) = \begin{cases} 1 & \text{for } -L \le x < 0 \\ 0 & \text{for } 0 \le x < L \end{cases}$ & $f(x) = f(x+2L)$ for all x



$$f(x) = a_0/2 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

(14) (cont)

where the coeffs are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 0$$

$$= \frac{1}{L}\int_{-L}^{0} 1\cos\left(\frac{n\pi x}{L}\right)dx + \cancel{\frac{1}{L}\int_{0}^{L} 0\cos\left(\frac{n\pi x}{L}\right)dx}^{\,0} \qquad n \geq 0$$

$$= \frac{1}{L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{0} \qquad n \geq 1$$

$$= 0 \qquad n \geq 1 \qquad (\sin(n\pi) = 0)$$

$$= \frac{1}{L}\int_{-L}^{0} dx \qquad n = 0$$

$$= 1 \qquad n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 1$$

$$= \frac{1}{L}\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{0}$$

$$= \frac{1}{n\pi}\left(-1 + (-1)^n\right)$$

$$f(x) = 1/2 + \sum_{1}^{\infty} \frac{1}{n\pi}\left(-1 + (-1)^n\right)\sin\left(\frac{n\pi x}{L}\right)$$

(NB: this agrees w/ the back of the book.)