p8 15 Let C = concentration of the chemical in the million gallon lake

Mass is conserved:

net buildup of chemical = rate in − rate out

$$\frac{d}{dt}\left(1000000\,C\right) = 0.01 \cdot 300 - C \cdot 300$$

$$\Rightarrow C' = 3\times10^{-6} - 3\times10^{-4}\,C$$

$$\Rightarrow C = (C_0 - 10^{-2})e^{-3\times10^{-4}t} + 10^{-2}$$

$\lim_{t\to\infty} C = 10^{-2}$ independent of C_0

p38 15 $ty' + 2y = t^2 - t + 1 \quad y(1) = 1/2 \quad t > 0$

We want to put this in standard form:

$$y' + \frac{2}{t}y = t - 1 + 1/t$$ (since $t \neq 0$, division by t is OK)

$$IF = \exp\left(\int \frac{2}{t}\,dt\right)$$

$$= \exp(2\log t)$$ (again, $t > 0$)

$$= t^2$$

$$\Rightarrow t^2y' + 2ty = t^3 - t^2 + t$$

$$\Rightarrow \frac{d}{dt}(t^2y) = t^3 - t^2 + t$$

$$\Rightarrow t^2y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$\Rightarrow y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$$

$y(1) = 1/2 \Rightarrow \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \Rightarrow C = 1/12$

p14 ⑥ $p = p(t)$ = field mouse population

$$\frac{dp}{dt} = 0.5p - 450$$

In this section, we find p by separation of variables (get the p's & dp's together, and the t's & the dt's):

$$\frac{dp}{0.5p-450} = dt$$

(so long as $p \neq 900$)
(if so $p(t) \equiv 900$ is a sol'n)

$$2\log|0.5p - 450| = t + C \qquad C \in \mathbb{R}$$

$$0.5p - 450 = De^{t/2}$$

$D \in \mathbb{R}$ (Note: we don't have to restrict $D > 0$ because we've dropped the absolute value bars; we can also include $D = 0$ to get the $p \equiv 900$ sol'n we excluded before)

$$\Rightarrow \quad p = 2De^{t/2} + 900$$

ⓐ If $p(0) = 850$, then

$$850 = 2De^{0/2} + 900$$

$$\Rightarrow \quad 2D = -50$$

$$\Rightarrow \quad p(t) = -50e^{t/2} + 900$$

If $p(t) = 0$, then $0 = -50e^{t/2} + 900$

$$\Rightarrow \quad e^{t/2} = 18$$

$$\Rightarrow \quad e = 2\log 18 \approx 5.8$$

ⓑ If $p(0) = p_0$ where $0 < p_0 < 900$, then

$$p_0 = 2De^{0/2} + 900$$

$$\Rightarrow \quad 2D = p_0 - 900 \quad \& \quad -900 < 2D < 0$$

$$\Rightarrow \quad p(t) = (p_0 - 900)e^{t/2} + 900$$

$$p(t) = 0 \Rightarrow 0 = (p_0 - 900)e^{t/2} + 900 \Rightarrow e^{t/2} = \frac{900}{900 - p_0}$$

$$e^{t/2} = \frac{900}{900-p_0} \qquad\qquad 0 < p_0 < 900$$

$$\Rightarrow 0 < 900 - p_0 < 900 \quad \left(\text{division is OK}\right)$$

$$\Rightarrow t/2 = \log\left(\frac{900}{900-p_0}\right) \qquad\qquad \Rightarrow 1 < \frac{900}{900-p_0} < +\infty \quad \left(\text{log is OK}\right)$$

$$\Rightarrow t = 2\log\left(\frac{900}{900-p_0}\right) \qquad\qquad \Rightarrow 0 < \log\left(\frac{900}{900-p_0}\right) < +\infty$$

(c) If $p(1) = 0$, then

$$0 = 2De^{1/2} + 900$$

$$\Rightarrow 2D = -\frac{900}{e^{1/2}}$$

$$\Rightarrow p(t) = -\frac{900}{e^{1/2}} e^{t/2} + 900$$

$$\Rightarrow p(0) = -\frac{900}{e^{1/2}} + 900 = 900\left(1 - e^{-1/2}\right) \approx 354$$

p22 (8) $y'' + 2y' - 3y = 0$

(a) $y_1(t) = e^{-3t}$

$y_1'(t) = -3e^{-3t}$

$y_1''(t) = 9e^{-3t}$

(b) $y_2(t) = e^t$

$y_2'(t) = e^t$

$y''(t) = e^t$

$$\forall t \in \mathbb{R} \quad 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0 \checkmark \qquad e^t + 2e^t - 3e^t = 0 \checkmark$$

(14) $y' - 2ty = 1$

$$y = \boxed{e^{t^2}} \int_0^t e^{-s^2}\,ds + e^{t^2}$$

$$y' = \left(2te^{t^2}\right)\left(\int_0^t e^{-s^2}\,ds\right) + \overbrace{\left(e^{t^2}\right)\left(e^{-t^2}\right)}^{1} + 2te^{t^2}$$

Use the product rule on □ & ○ and remember the Fund. Thm. of Calc.

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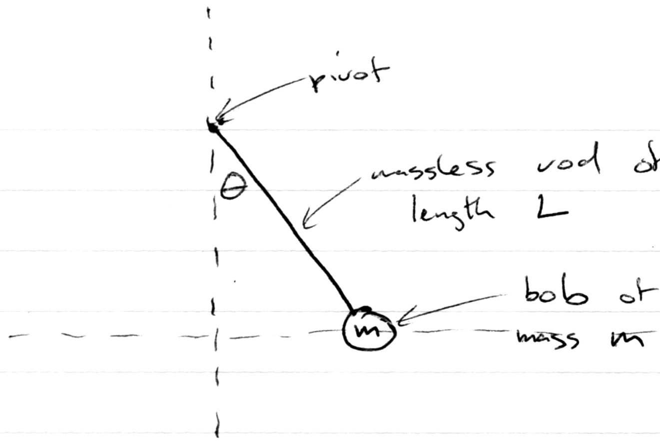

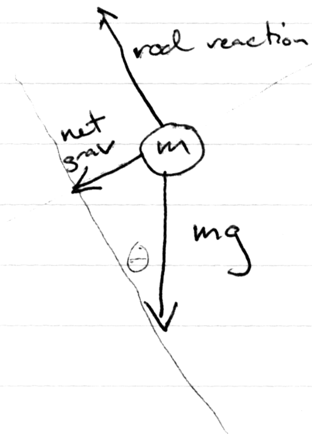

$$F = ma \quad \text{or} \quad ma = \sum F .$$

$$m \cdot L\ddot{\theta} = \text{net grav}$$

$$m \cdot L\ddot{\theta} = -mg \sin\theta \qquad (\theta \text{ is incr in a CCW fashion})$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta$$

If the displacement θ is small, then, to a good degree of approximation, $\sin\theta \approx \theta$ so that

$$\ddot{\theta} \cong -\frac{g}{L}\theta$$

$$\Rightarrow \theta \approx C_1 \sin\left(\sqrt{\tfrac{g}{L}}\,t\right) + C_2 \cos\left(\sqrt{\tfrac{g}{L}}\,t\right)$$

$$\| $$

$$D_1 \sin\left(\sqrt{\tfrac{g}{L}}\,t + D_2\right)$$

p38 (26) $y' + \frac{2}{3} y = 1 - t/2$

$$IF = \exp\left(\int \tfrac{2}{3}\,dt\right) = e^{2/3 t}$$

$$e^{2/3 t} y' + \tfrac{2}{3} e^{2/3 t} y = (1 - t/2) e^{2/3 t}$$

$$\frac{d}{dt}\left(e^{2/3 t} y\right) = (1 - t/2) e^{2/3 t}$$

$$e^{2/3 t} y = \int (1 - t/2) e^{2/3 t}\,dt$$

$$= \int e^{2/3 t}\,dt - \int \tfrac{t}{2} e^{2/3 t}\,dt$$

$$= \tfrac{3}{2} e^{2/3 t} - \left[\tfrac{3}{4} t e^{2/3 t} - \tfrac{9}{8} e^{2/3 t}\right] + C$$

$$\Rightarrow y = \tfrac{3}{2} - \tfrac{3}{4} t + \tfrac{9}{8} + C e^{-2/3 t}$$

We want to find C so that $((y \geq 0\ \forall t$ or $(y \leq 0\ \forall t))$ and $(\exists t_0$ s.t. $y(t_0) = 0)$. Thus we want the global min or the global max of y to be precisely 0.

$$y' = -\tfrac{3}{4} - \tfrac{2}{3} C e^{-2/3 t}$$

$$y' = 0 \Rightarrow -\tfrac{3}{4} - \tfrac{2}{3} C e^{-2/3 t} = 0 \Rightarrow e^{-2/3 t} = -\tfrac{9}{8C}$$ (if $C \neq 0$; however, note that $C=0$ gives a sol'n with no max or min so we lose nothing by excluding it

$$\Rightarrow -\tfrac{2}{3} t = \begin{cases} \log\left(-\tfrac{9}{8C}\right) & \text{if } C < 0 \\ \text{if } C > 0\text{, then there are no crit pts; that is, } y \text{ is always decreasing or always increasing; none of these are what we want} \end{cases}$$

$$\Rightarrow t = -\tfrac{3}{2} \log\left(-\tfrac{9}{8C}\right)$$

(26) (cont) We want this critical pt to be a max or a min (~~not~~ not a saddle point)

$$y'' = \frac{4}{9} C e^{-2/3 t}$$

$$y''(\text{crit pt}) = \frac{4}{9} C \exp\left(-\frac{2}{3} \cdot -\frac{3}{2} \log\left(-\frac{9}{8C}\right)\right)$$

$$= \left(\frac{4}{9} C\right)\left(-\frac{9}{8C}\right)$$

$$= -\frac{1}{2} \qquad (C \neq 0)$$

Thus (since $C \neq 0$) $y''(\text{crit pt}) < 0$ (not equal) so this is a max. It is the only max so it is the global max. We want

$$y(\text{global max}) = 0$$

$$\Rightarrow \frac{3}{2} - \frac{3}{4}\left(-\frac{3}{2} \log\left(-\frac{9}{8C}\right)\right) + \frac{9}{8} + C \exp\left(-\frac{2}{3} \cdot \frac{-3}{2} \log\left(-\frac{9}{8C}\right)\right) = 0$$

$$\Rightarrow \frac{21}{8} + \frac{9}{8} \log\left(-\frac{9}{8C}\right) - \frac{9}{8} = 0$$

$$\Rightarrow \frac{3}{2} = -\frac{9}{8} \log\left(-\frac{9}{8C}\right)$$

$$\Rightarrow -\frac{4}{3} = \log\left(-\frac{9}{8C}\right) \Rightarrow -\frac{9}{8C} = e^{-4/3}$$

$$\Rightarrow C = -\frac{9}{8} e^{4/3}$$

$$\Rightarrow y(0) = \frac{3}{2} - \frac{3}{4} \cdot 0 + \frac{9}{8} - \frac{9}{8} e^{4/3} e^{-2/3 \cdot 0}$$

$$= \frac{3}{2} + \frac{9}{8} - \frac{9}{8} e^{4/3} \approx -1.64$$

$\sin^2 x + \cos^2 x = 1$
$\cos(2x) = \cos^2 x - \sin^2 x$

p45 (5) $y' = (\cos^2 x)(\cos^2(2y))$

$$\Rightarrow \frac{dy}{\cos^2(2y)} = (\cos^2 x)\,dx$$

$$\Rightarrow \sec^2(2y)\,dy = \cos^2 x\,dx = \left(\frac{\cos(2x)+1}{2}\right)dx$$

$$\Rightarrow \frac{1}{2}\tan(2y) = \frac{1}{4}\sin(2x) + \frac{1}{2}x + C$$

$$\tan(2y) = \frac{1}{2}\sin(2x) + x + D$$

$$y = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\sin(2x) + x + D\right)$$

arctan is well def'd for all input arguments

(16) $y' = x(x^2+1)/4y^3$ $\quad y(0) = -2^{-1/2}$

y cannot be 0 here

$$\Rightarrow 4y^3\,dy = x(x^2+1)\,dx$$

$$\Rightarrow y^4 = \underbrace{\frac{x^4}{4} + \frac{x^2}{2}}_{\text{always } \geq 0} + C$$

$$\Rightarrow y = \pm\left(x^4/4 + x^2/2 + C\right)^{1/4}$$

choose the sign to agree w/ the IC.

domain is all x if $C \geq 0$

if $C < 0$, then the domain is all $x \geq \left(-1+\sqrt{1-4C}\right)^{1/2}$

and all $x \leq -\left(-1+\sqrt{1-4C}\right)^{1/2}$

there is non-uniqueness in the sol'n!

for $y(0) = -2^{-1/2}$, $y = -\left(x^4/4 + x^2/2 + 1/4\right)^{1/4}$ $\forall x \in \mathbb{R}$

In general: net rate = rate in − rate out

p58 (10) M = amt left due on mortgage

$$M' = \frac{0.09}{12} M - 800 \qquad (t \text{ in months})$$

$$\Rightarrow M = \left(M_0 - \frac{12 \cdot 800}{0.09}\right) e^{\frac{0.09}{12}t} + \frac{12 \cdot 800}{0.09}$$

in order for $M' < 0$ (i.e. the amt left decreases with time so that the loan is eventuall paid off) we need:

$$M' = \frac{0.09}{12}\left(M_0 - \frac{12 \cdot 800}{0.09}\right) e^{\frac{0.09}{12}t} < 0$$

$$\Rightarrow M_0 - \frac{12 \cdot 800}{0.09} < 0 \Rightarrow M_0 < 166{,}666\tfrac{2}{3}$$

The loan is paid off when $M = 0$

(assuming that the loan can be paid off: $M_0 < 166{,}666\tfrac{2}{3}$)

$$\Rightarrow 0 = \left(M_0 - \frac{12 \cdot 800}{0.09}\right) e^{\frac{0.09}{12}t} + \frac{12 \cdot 800}{0.09}$$

$$\Rightarrow \frac{\frac{12 \cdot 800}{0.09}}{\frac{12 \cdot 800}{0.09} - M_0} = e^{\frac{0.09}{12}t} \Rightarrow \frac{0.09}{12} t = \log\left(\frac{320000}{320000 - 3M_0}\right)$$

$$\Rightarrow t = \frac{12}{0.09} \log\left(\frac{320000}{320000 - 3M_0}\right)$$

The amount paid out is 800 times this time:

$$800 \cdot \frac{12}{0.09} \log\left(\frac{320000}{320000 - 3M_0}\right) = \frac{320000}{3} \log\left(\frac{320000}{320000 - 3M_0}\right)$$

p59 (15) $p = p(t) = $ mosquito population

$$p' = \left(\frac{\text{\# of new mosiquitos bred}}{\text{day}} \right) - \left(\frac{\text{\# eaten}}{\text{day}} \right)$$

$$= rp - 20000$$

$$r = (\log 2)/\text{week} = (7 \log 2)/\text{day}$$

$$p = \left(p_0 - \frac{20000}{r}\right)e^{rt} + \frac{20000}{r}$$

Since $p_0 = 200,000$ and

$$p_0 - \frac{20000}{r} = 20000\left(10 - \frac{1}{r}\right)$$

$$= 20000\left(10 - \frac{1}{7\log 2}\right) > 0,$$

$p \to +\infty$ as $t \to +\infty$