(1) Series solutions can be used to solve 2^{ND} order & higher order linear ODEs (we will be focussing on the homogeneous case only).

(2) In order to apply series solutions to an ODE, a center point for that series needs to be chosen (according to the dictates of the physical process being modelled). That center point for the series is either going to be a singular point of the ODE (a point where the coefficient of the highest order term is 0 or where the coefficient of any other term blows up — we won't be considering any other types of discontinuities), or an ordinary point (everywhere else). For example,

$$(x^2-2x-3)y'' + (\log|x|)y' + (\sin x)y = 0$$

has singular points at $x = -1, 0,$ and 3 and regular points everywhere else (p238)

(3) For series about an ordinary point, you should be able to compute the coefficients of the series when the coefficients of the ODE are polynomials (for example, in the case:

$$(x^3-6x)y'' + (x+5)y' + (x^4+x-2)y = 0\ )$$

That is, substitute the series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and its derivatives into the ODE, collect terms of like order in x $(1, x, x^2, x^3, \ldots)$, and match coefficients of x^n between the LHS & RHS of the ODE to get a recursion relation for the a_n's. (p238-247)

You should also know that the radius of convergence of the above series is at least as large as the minimum of (a) the distances from x_0 to the zeros of $P(x)$, the coefficient of the highest order term, y'', in the ODE, and (b) the distances from x_0 to the poles of all the other coefficients in the ODE. (p250-253)
(That is, the distances from x_0 to the singular pts of the ODE)

Lastly, if we are given the ODE

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = 0$$

and an ordinary pt x_0 of the ODE (so $P(x_0) \neq 0$ and $Q(x)$ & $R(x)$ are well-behaved near x_0), then we can rearrange it to:

$$y''\cancel{(x)} = -\,p(x)\, y'(x) - q(x)\, y(x)$$

This gives a short-cut to the first few terms of the series sol'n (p249).

(4) If you wish to obtain a series solution about a singular point of the ODE, you need to be able to tell whether the singular point is a regular one (which we'll show how to do) or an irregular one (which would take far too much time to do in this class, but can be found in other textbooks). If we have the ODE: $P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = 0$, the point x_0 is a regular singular point if both

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}$$

and

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

exist and are finite. (p 255–259)

For example, the Euler equation:

$$x^2 y'' + \alpha x y' + \beta y = 0$$

has ordinary points for $x_0 \neq 0$ and a regular singular point at $x = 0$ because:

$$\lim_{x \to 0} (x-0)\frac{\alpha x}{x^2} = \alpha \quad \text{and} \quad \lim_{x \to 0} (x-0)^2 \frac{\beta}{x^2} = \beta$$

35.5

(5) For the Euler equation (p260-265):

$$x^2 y'' + \alpha x y' + \beta y = 0$$

There is a regular singular point at $x=0$. We guess $y = x^r$ is a sol'n to get the indicial equation: $r(r-1) + \alpha r + \beta = 0$

(kind of like the characteristic equation for 2^{ND} order, linear, homogeneous equations w/ $y = e^{rx}$).

Case 1: distinct real roots r_1 & r_2 of the indicial eqn:

$$y = C_1 x^{r_1} + C_2 x^{r_2}$$

Case 2: a repeated real root r_1:

$$y = C_1 x^{r_1} + C_2 x^{r_1} \log x$$

Case 3: complex roots $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$:

$$y = C_1 x^{\lambda} \cos(\mu \log x) + C_2 x^{\lambda} \sin(\mu \log x)$$

(6) For any other equation with a regular singular point x_0, we'll try to massage it to make it look like the Euler equation, use the indicial equation that we get for the Euler-like equation, and then use the solutions we get from the indicial equation multiplied by a series to get solutions of

The desired problem.

(see p 267-271 & for an intro, p 272-278 for a more detailed discussion, and the blue box on p 277-278 for a summary)

Remember: exam questions come from the homework! Practice with the assigned problems. They break this method into pieces for you to practice each part separately.

Also note that one can always use this method to obtain a series rep. ~~for~~ associated with the larger root of the indicial eqn. There are different cases (several) for finding the solution assoc. w/ the smaller root. You could forget all this & just use reduction of order w/ the sol'n you've found to find the one you didn't get.