Say I want to find a series solution for

$$x^2 y'' - x(x+2) y' + (x^2+2) y = 0$$

around $x=0$. However $x=0$ is a singular point ($0^2 = 0$ but ~~$0(0+2) \neq 0$ / [illegible]~~ $0^2+2=2$). Thus a guess of $y(x) = \sum_{n=0}^{\infty} a_n x^n$ won't work. (Try it!) Also, $x=0$ is a regular singular point because:

$$\lim_{x \to 0} (x-0) \frac{(-1)x(x+2)}{x^2} = -2 \qquad \text{(finite!)}$$

$$\lim_{x \to 0} (x-0)^2 \frac{(x^2+2)}{x^2} = 2 \qquad \text{(finite!)}$$

so the singularity isn't too bad.

The Euler equation

$$x^2 y'' + \alpha x y' + \beta y = 0$$

is sort-of a canonical representative of 2^{ND} order, linear, homogeneous ODEs w/ a singular point at $x=0$. I'm going to try to make our problem look like this one.

$$x^2y'' - x(x+2)y' + (x^2+2)y = 0$$

$$\Downarrow$$

$$x^2y'' + (-2x - \cancel{x^2})y' + (2 + \cancel{x^2})y = 0$$

small compared to other terms in the equation when $x \approx 0$

$$\approx$$

$$x^2y'' - 2xy' + 2y = 0$$

Since this is an Euler eqn, I can guess $y = x^r$ & find the indicial equation:

$$r(r-1) - 2r + 2 = 0$$

$$\Rightarrow r^2 - 3r + 2 = 0$$

$$\Rightarrow (r-1)(r-2) = 0$$

$$\Rightarrow r=1 \text{ or } r=2$$

I now take the larger of these two roots, $r=2$, and use it to patch up my series sol'n guess:

$$y = x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^{n+2} = a_0 x^2 + a_1 x^3 + \ldots$$

$$y' = \sum_{n=0}^{\infty} (a_n)(n+2) x^{n+1} = 2a_0 x + 3a_1 x^2 + \ldots$$

$$y'' = \sum_{n=0}^{\infty} (a_n)(n+2)(n+1) x^n = 2a_0 + 6a_1 x + \ldots$$

so $x^2 y'' - x(x+2) y' + (x^2+2) y = 0$

implies $\displaystyle\sum_{n=0}^{\infty} (a_n)(n+2)(n+1) x^{n+2} - \sum_{n=0}^{\infty} (a_n)(n+2) x^{n+3}$

$$- \sum_{n=0}^{\infty} (2a_n)(n+2) x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+4} + \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0$$

The last two begin w/ x^4. All the others begin with lower powers of x so lets coerce them all to that form.

$$\sum_{n=0}^{\infty}(a_n)(n+2)(n+1)x^{n+2} = 2a_0x^2 + 6a_1x^3 + \sum_{n=2}^{\infty}(a_n)(n+2)(n+1)x^{n+2}$$

$$m+4 = n+2$$

$$= 2a_0x^2 + 6a_1x^3 + \sum_{m=0}^{\infty}(a_{m+2})(m+4)(m+3)x^{m+4}$$

$$\sum_{n=0}^{\infty}(a_n)(n+2)x^{n+3} = 2a_0x^3 + \sum_{n=1}^{\infty}(a_n)(n+2)x^{n+3}$$

$$= 2a_0x^3 + \sum_{m=0}^{\infty}(a_{m+1})(m+3)x^{m+4}$$

$$\sum_{n=0}^{\infty}(2a_n)(n+2)x^{n+2} = 4a_0x^2 + 6a_1x^3 + \sum_{n=2}^{\infty}(2a_n)(n+2)x^{n+2}$$

$$= 4a_0x^2 + 6a_1x^3 + \sum_{m=0}^{\infty}(2a_{m+2})(m+4)x^{m+4}$$

$$\sum_{n=0}^{\infty} a_n x^{n+4} \quad \text{OK}$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+2} = 2a_0x^2 + 2a_1x^3 + \sum_{n=2}^{\infty} 2a_n x^{n+2}$$

$$= 2a_0x^2 + 2a_1x^3 + \sum_{m=0}^{\infty}(2a_{m+2})(x^{n+4})$$

Collecting like terms when putting this together gives:

$$(2a_1 - 2a_0)x^3 + \sum_{m=0}^{\infty}\Big[(a_{n+2})(n+4)(n+3) - (a_{n+1})(n+3) - (2a_{n+2})(n+4) + a_n + 2a_{n+2}\Big]x^{m+4} = 0$$

Matching coefficients of x from the LHS & RHS:

$$2a_1 - 2a_0 = 0$$

$$(a_{n+2})(n+4)(n+3) - (a_{n+1})(n+3) - (2a_{n+2})(n+4) + a_n + 2a_{n+2} = 0$$

$$a_1 = a_0$$

$$(n^2+5n+6)a_{n+2} - (n+3)a_{n+1} + a_n = 0$$

There's only one free parameter, a_0!

(and an infinite radius of convergence!)

We expect this because we've only found one independent sol'n of the ODE. To find the other either requires us to use ① three different methods depending on whether:

(a) $r_1 - r_2$ is not an integer (where r_1 & r_2 are the roots of our indicial eqn)

(b) $r_1 = r_2$

(c) $r_1 - r_2$ is an integer

(in order of increasing complexity)

Or ② the method of reduction of order

(we have one sol'n of a 2^{ND} order ODE; we can use it to find a 1^{ST} order ODE for the second sol'n)

Ex 2

Say I want to find a series sol'n for

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0$$

around $x = 3$. This is a singular pt:

$3^2 - 2\cdot 3 - 3 = 0$ but 3 & 4 are finite & non-zero.

It is a regular singular point because:

$$\lim_{x \to 3} (x-3)\frac{x}{(x^2-2x-3)} = \lim_{x \to 3} \frac{x}{x+1} = \frac{3}{4} \quad \text{(finite)}$$

$$\lim_{x \to 3} (x-3)^2 \frac{4}{(x^2-2x-3)} = \lim_{x \to 3} \frac{(x-3)(4)}{x+1} = 0 \quad \text{(finite)}$$

The Euler eqn around $x = 3$ is:

$$(x-3)^2 y'' + \alpha (x-3) y' + \beta y = 0$$

Our eqn is

$$(x-3)(x+1) y'' + (x) y' + 4y = 0$$

or

$$(x-3)^2 y'' + \frac{(x)(x-3)}{x+1} y' + \frac{4(x-3)}{x+1} y = 0$$

We need to expand $p(x) = \dfrac{x}{x+1}$ and $q(x) = \dfrac{4(x-3)}{x+1}$ around $x = 3$. One can either use Taylor's Theorem, or a creative use of power series:

$$\frac{x}{x+1} = 1 - \frac{1}{x+1} = 1 - \frac{1}{4}\left(\frac{1}{1+\left(\frac{x-3}{4}\right)}\right)$$

$$= 1 - \frac{1}{4}\left(1 - \left(\frac{x-3}{4}\right) + \left(\frac{x-3}{4}\right)^2 - \cdots\right)$$

(so long as $\left|\frac{x-3}{4}\right| < 1$)

$$= \frac{3}{4} + \frac{1}{16}(x-3) - \frac{1}{64}(x-3)^2 + \cdots \quad (|x-3| < 4)$$

$$\frac{4(x-3)}{x+1} = 4(x-3)\left(\frac{1}{1+x}\right) = 4(x-3)\,\frac{1}{4}\left(\frac{1}{1+(x-3/4)}\right)$$

$$= (x-3)\left(1 - \left(\frac{x-3}{4}\right) + \left(\frac{x-3}{4}\right)^2 - \cdots\right) \quad \left(\left|\frac{x-3}{4}\right| < 1\right)$$

$$= (x-3) - \frac{1}{4}(x-3)^2 + \frac{1}{16}(x-3)^3 - \cdots \quad (|x-3| < 4)$$

All we really care about are the constant terms:

$$\underbrace{(x-3)^2 y'' + \left(\frac{3}{4} + \frac{1}{16}(x-3) - \frac{1}{16}(x-3)^2 + \dots\right)(x-3)y' + \left(0 + (x-3) - \frac{1}{4}(x-3)^2 + \dots\right) y = 0}_{\approx\ (x-3)^2 y'' + \alpha (x-3) y' + \beta y}$$

$$(x-3)^2 y'' + \frac{3}{4}(x-3)y' + 0y = 0 \quad \text{when } x \approx 3$$

(this $\frac{3}{4}$ & 0 are also the same as the results of the limits we computed when determining whether $x=3$ was a singular pt; this is no small coincidence – it'll always be true)

We now guess $y = (x-3)^r$ to get the indicial equation: $r(r-1) + \frac{3}{4}r + 0 = 0$

or $r^2 - \frac{1}{4}r = 0$

so $r = \frac{1}{4}$ or $r = 0$.

We take the larger root, $r = 1/4$, and guess

$$y = x^{1/4} \sum_{n=0}^{\infty} a_n (x-3)^n$$

(this has a radius of convergence of at least 4 around $x=3$)

in the original ODE $(x^2-2x-3)y'' + xy' + 4y = 0$