1

$$2xy'' + y' + xy = 0 \quad \text{around} \quad x = 0$$

$$y'' + \frac{1}{2x} y' + \frac{x}{2x} y = 0$$

$$x^2 y'' + \frac{x^2}{2x} y' + \frac{x^3}{2x} y = 0$$

$$x^2 y'' + \left(\frac{x}{2x}\right) x y' + \left(\frac{x^3}{2x}\right) y = 0$$

$$\lim_{x \to 0} \frac{x}{2x} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{x^3}{2x} = \boxed{0}$$

$$\frac{x}{2x} \text{ "=" } \frac{1}{2} + 0x + 0x^2 + 0x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} p_n x^n$$

$$\frac{x^3}{2x} \text{ "=" } \boxed{0} + 0x + \frac{1}{2}x^2 + 0x^3 + 0x^4 + \cdots$$

$$= \sum_{n=0}^{\infty} q_n x^n$$

The indicial equation is:

$$r(r-1) + \frac{1}{2}r + \boxed{0} = 0$$

$$\Rightarrow \quad r(r - 1/2) = 0$$

$$\Rightarrow \quad r = 0 \text{ or } 1/2$$

Also $F(r) = r(r-1) + \frac{1}{2}r + 0$

$$= r(r - 1/2)$$

So that the recurrence relation is:

$$F(r+n)\,a_n + \sum_{k=0}^{n-1} a_k\left[(r+k)p_{n-k} + q_{n-k}\right] = 0 \qquad \forall n \geq 1$$

where F is as above (the LHS of the indicial eqn), r is one of the roots of $F(r)=0$, $\{p_n\}_{n=0}^{\infty}$ & $\{q_n\}_{n=0}^{\infty}$ are the coeffs. of the series rep. for $\frac{x}{2x}$ & $\frac{x^3}{2x}$ respectively, as above.

Our series look like:

$$y_i = \underbrace{|x-0|}_{\text{signum}(x-0)}{}^{r_i} \sum_{n=0}^{\infty} a_{i,n}(x-0)^n$$

For $r=0$ the recurrence relation is:

$$\boxed{n(n-1/2)\,a_n + \frac{1}{2}a_{n-2} = 0 \quad \forall n \geq 2} \quad \& \quad \boxed{\frac{1}{2}a_1 + 0 = 0 \quad (n=1)}$$

$$\Rightarrow \quad a_n = -\frac{a_{n-2}}{n(2n-1)} \quad \forall n \geq 2 \quad \& \quad a_1 = 0$$

Thus the solution corresponding to $r=0$ is:

$$y_1 = 1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \frac{1}{11088}x^6 + \dots$$

For $r=1/2$ the recurrence relation is:

$$\boxed{(n+1/2)\,n\,a_n + \frac{1}{2}a_{n-2} = 0 \quad \forall n \geq 2} \quad \& \quad \boxed{\frac{3}{2}a_1 + 0 = 0 \quad (n=1)}$$

$$\Rightarrow \quad a_n = -\frac{a_{n-2}}{(2n+1)n} \quad \forall n \geq 2 \quad \& \quad a_1 = 0$$

Thus the solution corresponding to $r=1/2$ is:

$$y_2 = \underset{\text{signum}(x)}{\vee}\,|x|^{1/2}\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \frac{1}{28080}x^6 + \dots\right)$$

The solution to the ODE is $y = C_1 y_1 + C_2 y_2$

where C_1 & C_2 are arbitrary constants.

Note: to get the recurrence relation in this problem, you could follow the familiar (from ordinary point theory):

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1/2}$$

$$y_1' = \sum_{n=0}^{\infty} a_n (n+1/2) x^{n-1/2}$$

...

followed by subbing in the DE and matching terms of like order.

The formula in the book $\left(F(r+n)a_n + \sum \ldots \right\} = 0\right)$ has done all that for you; it may look imposing, though.

②

$$(\log x)\, y'' + \tfrac{1}{2} y' + y = 0 \qquad \text{around} \quad x=1$$

$$y'' + \frac{1}{2\log x} y' + \frac{1}{\log x} y = 0$$

$$(x-1)^2 y'' + \left(\frac{x-1}{2\log x}\right)(x-1)\, y' + \left(\frac{(x-1)^2}{\log x}\right) y = 0$$

$$\lim_{x\to 1} \frac{x-1}{2\log x} = \lim_{x\to 1} \frac{1}{2/x} = \frac{1}{2}$$

$$\lim_{x\to 1} \frac{(x-1)^2}{\log x} = \lim_{x\to 1} \frac{2(x-1)}{1/x} = \boxed{0}$$

$$\frac{x-1}{2\log x} = \frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{24}(x-1)^2 + \frac{1}{48}(x-1)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} p_n (x-1)^n$$

$$\frac{(x-1)^2}{\log x} = \boxed{0} + 1(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{12}(x-1)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} q_n (x-1)^n$$

Since $\log 1 = 0$, and "$1/2$" & "1" are well-behaved near $x=1$, we have that $x=1$ is a singular point of the ODE. Moreover, the above limits exist and are finite, so $x=1$ is a regular singular point.

The indicial equation for the exponents at the singularity is:

$$r(r-1) + \frac{1}{2} r + \boxed{0} = 0$$

$$\Rightarrow \quad r(r-1/2) = 0$$

$$\Rightarrow \quad r = 0 \text{ or } 1/2$$

Also $F(r) = r(r-1) + \frac{1}{2} r + 0$
$= r(r - 1/2)$

We need only find the series solution corresponding to $r = 1/2$, the larger root.

Our recurrence relation is:

$$F(1/2 + n)\, a_n + \sum_{k=0}^{n-1} a_k \left[(1/2 + k) p_{n-k} + q_{n-k} \right] = 0 \quad \forall n \geq 1$$

where the symbols are defined similarly to ①.

Our series looks like $y_1 = {}^{v}|x-1|^{1/2} \sum_{n=0}^{\infty} a_n (x-1)^n$

signum$(x-1)$

Take a_0 as given. Then for $n=1$:

$$F(3/2)\, a_1 + \sum_{k=0}^{0} a_k \left[(1/2 + k)\, p_{1-k} + q_{1-k} \right] = 0$$

$$F(3/2)\, a_1 + a_0 \left[\tfrac{1}{2} p_1 + q_1 \right] = 0$$

$$\frac{3}{2} a_1 + \left(\frac{1}{2} \cdot \frac{1}{4} + 1 \right) a_0 = 0$$

$$a_1 = -\frac{2}{3} \cdot \frac{9}{8} a_0 = -\frac{3}{4} a_0$$

For $n=2$:

$$F(5/2)\, a_2 + \sum_{k=0}^{1} a_k \left[(\tfrac{1}{2} + k)\, p_{2-k} + q_{2-k} \right] = 0$$

$$F(5/2)\, a_2 + a_0 \left[\tfrac{1}{2} p_2 + q_2 \right] + a_1 \left[\tfrac{3}{2} p_1 + q_1 \right] = 0$$

$$5a_2 + \left(\frac{1}{2}\left(-\frac{1}{24}\right) + \frac{1}{2} \right) a_0 + \left(\frac{3}{2} \cdot \frac{1}{4} + 1 \right)\left(-\frac{3}{4} a_0 \right) = 0$$

$$5a_2 + \frac{23}{48} a_0 - \frac{33}{32} a_0 = 0$$

$$5a_2 - \frac{53}{96} a_0 = 0 \implies a_2 = \frac{53}{480} a_0$$

Thus $y_1 = \underset{\text{signum}(x-1)}{\vee} |x-1|^{1/2} \left(1 - \frac{3}{4}(x-1) + \frac{53}{480}(x-1)^2 + \cdots \right)$

The ODE also has a singular point at $x=0$ (and no others). Thus we would expect the above series to have a radius of convergence of at least $|1-0| = 1$.

$$\underset{\substack{\uparrow \\ \text{center of our series}}}{1} - \underset{\substack{\uparrow \\ \text{only other singular pt}}}{0}$$

Note: to get the recurrence relation this time, you could take $y_2 = \sum_{n=0}^{\infty} a_n x^{n+1/2}$ and sub in the DE again, but you'll also have to get series for $\frac{x-1}{2\log x}$ & $\frac{(x-1)^2}{\log x}$ to complete the substitution. Then you'll need to <u>multiply</u> <u>two</u> <u>series</u> <u>together</u>. Again, the formula in the book has already done all this for you.

If you assume $x > 0$ (in the first problem) or $x > 1$ (in the second problem) as you'll be allowed to do on the test, ignore the absolute value bars and the signum stuff.