The following represents some of what is possible (not necessarily what is expected). Your mileage may vary.

Group Zero HW Project One

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Drawing a direction field and performing analysis with a pen and paper has many advantages. One can obtain clear insight into the behavior of the sol'n to an IVP with some simple algebra & calculus. Exact answers are possible. One need not learn the syntax & quirks of a software package.

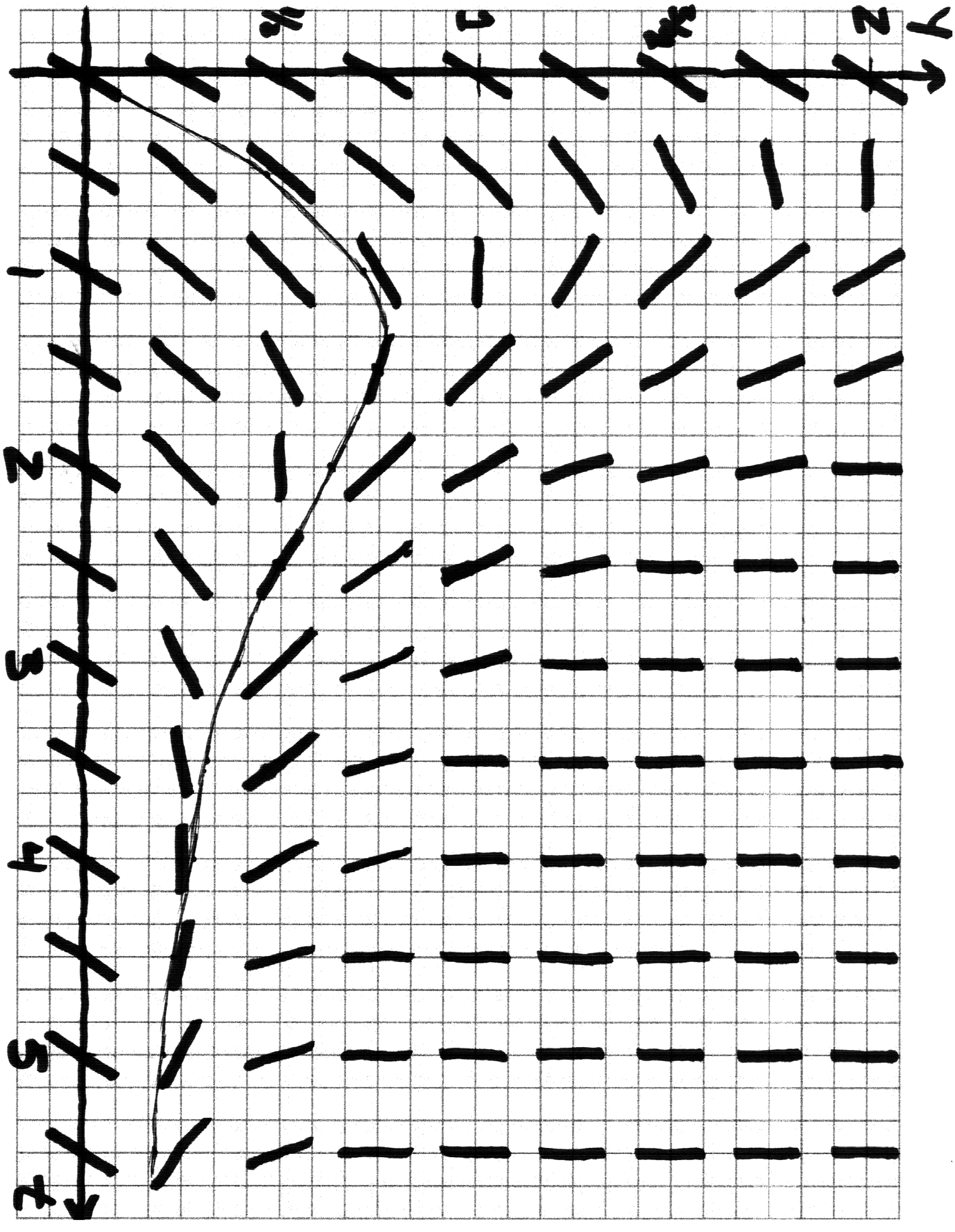
However, drawing a direction field by hand can be tedious & slow. Some problems (especially non-linear ones) are not amenable to easy analysis.

We were able to conclude that:

1. $y(4) = e^{-8}\int_0^4 e^{s^2/2}\,ds \approx 0.2704$

and

2. $\lim y = 0$.



Analysis / Support:

$y' = 1 - ty$

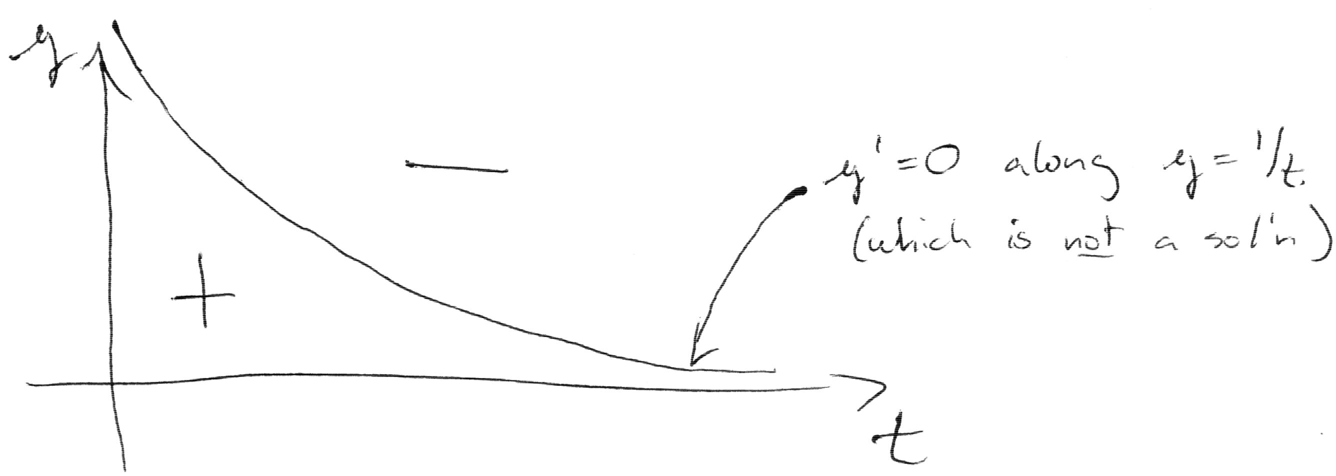
$y(0) = 0$

$\Rightarrow y'(0) = 1$

$\Rightarrow$ the sol'n graph enters the 1st quad.

Since $y' \equiv 1$ along the curve $y = 0$, once a sol'n curve enters the 1st quad, it cannot leave. Thus the sol'n to our IVP ~~as~~ has $y > 0$ for $t > 0$. Since we're asked for the limit as $t \to \infty$ but not for $t \to -\infty$, we'll only plot $t \geq 0$.

Schematic of the sign of y':



The schematic suggests the following:

① Any sol'n curve in the first quadrant crosses the curve $y = 1/t$ exactly once.

If the sol'n is below $1/t$, it must be increasing whereas $1/t$ is decreasing.

If the sol'n is above $1/t$, it cannot cross $1/t$ again since

$$\underbrace{0}_{\text{slope of sol'n along } y=1/t} > \underbrace{-2/t^2}_{\text{slope of } y=1/t} \quad (\text{for } t>0)$$

② The sol'n y to our IVP obeys $y < t$ for $t<0$ since

$$\underbrace{1-t^2}_{\text{slope } y' \text{ along } y=t} < \underbrace{1}_{\text{slope of } y=t} \quad \text{for } t>0$$

~~Since~~ Furthermore, since the sol'n y reaches it's max where it intersects $1/t$ (where $y'=0$ & y' changes from + to −), the max of our sol'n must occur to the right of $t=1$ & below $y=1$.

Interlude:

Since we're in the first quad & y's max is <1, we'll pick $0 \le y \le 2$ for the graph. Since we want $y(4)$ & are given $y(0)$, we'll use $0 \le t \le 5$.

What follows below is almost all gravy (but does help w/ that upper bound on t & evaluating $y(4)$).

The schematic further suggests:

(3) Along w/ (1) that $\lim_{t\to\infty} y = 0$. To prove it,

use the same ideas:

for $t > \sqrt{2}$ & $y \geq 2/t$ then $\underset{\text{slope of sol'n}}{y'} \leq -1 < \underset{\text{slope of } 2/t}{-\frac{2}{t^2}}$

Thus the sol'n will eventually cross $2/t$ & thereafter be bdd above by $2/t$.

for all t & $y = 0$ then $\underset{\text{slope of sol'n}}{y'} = 1 > \underset{\text{slope of } y=0}{0}$

Thus the sol'n (once it crosses $y=0$) will thereafter stay above 0. Thus for some t_0

$0 < y < 2/t$ for $t > t_0$ so

$$0 = \lim_{t\to\infty} 0 \leq \lim_{t\to\infty} y \leq \lim_{t\to\infty} \frac{2}{t} = 0$$

so that $\lim_{t\to\infty} y = 0$.

We also showed that $1/t < y$ for t sufficiently large (in (1)). We can repeat the above w/ any $C > 1$ & $y \geq C/t$ & $t > \sqrt{\frac{C}{C-1}}$ to get $y' \leq 1 - C < -C/t^2$ so that $\frac{1}{t} < y < \frac{C}{t}$ for large t and <u>any</u> $C > 1$. Thus the sol'n

y is asymptotic to $1/t$ $\left(\lim_{t\to\infty} \frac{y}{1/t} = 1 \right)$.

Computing the sol'n more explicitly verifies all the above:

$$\left.\begin{aligned} y' &= 1 - ty \\ y(0) &= 0 \end{aligned}\right\} \Rightarrow y = e^{-t^2/2} \int_0^t e^{s^2/2}\, ds$$

Since $e^{-t^2/2}$ & $e^{s^2/2}$ are always positive, then so is $y > 0$. To find the max in y, compute:

$$y' = 1 - ty$$
$$y'' = (t^2 - 1)y - t$$

Then use a couple of iterations of Newton's method along with a few steps of adaptive Simpson's rule to find that $y' \doteq 0$ for $t \simeq 1.3$. Then $y(1.3) \doteq 0.77$ (which is $\doteq 1/1.3$). Note that $0.77 < 1 < 1.3$ as predicted.

Using L'Hopital's rule, we can check that $\lim y = 0$, $\lim y/\tfrac{1}{t} = 1$. (We can also further compute $\lim \frac{y - 1/t}{1/t^2} = 0$ and $\lim \frac{y - 1/t}{1/t^3} = 1$ to get that $y = \frac{1}{t} + \frac{1}{t^3} + O\left(\frac{1}{t^4}\right)$ as $t \to \infty$.)

Lastly, one can again use Simpson's adaptive quadrature to approximate:

$$y(4) = e^{-8} \int_0^4 e^{s^2/2} \, ds \approx 0.2704$$

This agrees well with the asymptotics

$$y(4) \approx 1/4 \qquad \left(\text{and } y(4) \approx \frac{1}{4} + \frac{1}{64} = 0.265625\right)$$

All of these results agree with the graph estimated from the direction field.