Test 1, 427K, February 26 2004 **PRINTED** NAME: WHITE

Every problem is worth an equal number of points for a total of 100 points. You must show your work; answers without substantiation do not count. Answers must appear in the box provided.

(I) Find the general solution of FOLODE $y' + 2ty = 2te^{-t^2}$.

$$IF = \exp\left(\int 2t\,dt\right) = e^{t^2}$$

$$\left(y e^{t^2}\right)' = 2t$$

$$y e^{t^2} = t^2 + C$$

Answer: $y = (t^2 + C)e^{-t^2}$

(II) Find an integrating factor μ and solve implicitly $e^x\,dx + (e^x \cot y + 2y \csc y)\,dy = 0$.

$$0 \neq e^x \cot y$$

$$\mu_y \cdot e^x + \mu \cdot 0 = \mu_x(e^x \cot y + 2y \csc y) + \mu \cdot e^x \cot y$$

$$\mu = \mu(y) \Rightarrow \mu_x = 0 \ \& \ \mu_y = \mu'$$

$$\mu' e^x = \mu e^x \cot y \Rightarrow \frac{\mu'}{\mu} = \cot y \Rightarrow \log|\mu| = \log|\sin y| + C$$

$$\mu = D \sin y$$

$$e^x \sin y\,dx + (e^x \cos y + 2y)\,dy = 0$$

$$\Psi_x\,dx + \Psi_y\,dy$$

$$\Psi = e^x \sin y + H(y)$$

$$\Psi_y = e^x \cos y + H'(y) \Rightarrow H'(y) = 2y$$

Answer: $\mu(y) = \sin y$ solution: $e^x \sin y + y^2 = c$

(III) State the **general** existence and uniqueness theorem for the **general** first order initial value problem.

S'pose that

IVP

① $y' = f(t,y)$

② f & $\frac{\partial f}{\partial y}$ are continuous in some region R

(R must be open and simply connected)

③ $y(t_0) = y_0$ and $(t_0, y_0) \in R$

then there exists a unique sol'n to the IVP (so long as $(t, y(t)) \in R$).

(IV) The differential equation $x^2y'' + 2xy' - 2y = 0$ has for $x > 0$ the solution $y_1(x) = x$. Find a second solution by the method of reduction of order.

$$-2\left(y_2 = vx\right)$$
$$+2x\left(y_2' = v'x + v\cdot 1\right)$$
$$+x^2\left(y_2'' = v''x + 2v'\cdot 1 + v\cdot 0\right)$$

$$0 = 2v'\cdot x^2 + v''\cdot x^3 + 2v'\cdot x^2$$

$$0 = x\cdot v'' + 4v' \qquad (x>0)$$

$$0 = v'' + \frac{4}{x}v'$$

$$IF = \exp\left(\int \frac{4}{x}\,dx\right) = x^4$$

$$0 = \left(v'\cdot x^4\right)'$$

$$C = v'x^4$$

$$Cx^{-4} = v'$$

$$-\frac{C}{3}x^{-3} + D = v$$

$$y_2 = \left(-\frac{C}{3}x^{-3} + D\right)x = \tilde{C}x^{-2} + Dx$$

Answer: $\tilde{y}_2(x) =$ x^{-2}

(V) Use the method of Variation of Parameters to find a particular solution $Y(t)$ of the equation $(1-t)y''+ty'-y = 2(t-1)^2e^{-t}$, $0 < t < 1$, given that $y_1(t) = t$ and $y_2(t) = e^t$ form a fundamental system for the homogeneous equation.

$$y_H = C_1 y_1 + C_2 y_2$$

$$Y = y_I = u_1 y_1 + u_2 y_2$$

$$y'' + \frac{t}{1-t} - \frac{1}{1-t} y = -2(t-1)e^{-t}$$

$$u_1 = -\int \frac{RHS \cdot y_2}{W(y_1, y_2)}\, dt$$

$$u_2 = +\int \frac{RHS \cdot y_1}{W(y_1, y_2)}\, dt$$

$$W = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = (t-1)e^t$$

$$u_1 = -\int \frac{-2(t-1)e^{-t} \cdot e^t}{(t-1)e^t}\, dt$$

$$= \int 2e^{-t}\, dt = -2e^{-t}$$

$$u_2 = \int \frac{-2(t-1)e^{-t} \cdot t}{(t-1)e^{-t}}\, dt$$

$$= \int -2te^{-2t}\, dt = -\frac{1}{2}\int ue^{u}\, du$$

$$= -\frac{1}{2}(u-1)e^{u} = -\frac{1}{2}(-2t-1)e^{-2t}$$

Answer: $Y(t) = -2te^{-t} + \frac{1}{2}(2t+1)e^{-t}$

Test 1, 427K, February 26 2004 **PRINTED** NAME: YELLOW

Every problem is worth an equal number of points for a total of 100 points. You must show your work; answers without substantiation do not count. Answers must appear in the box provided.

(I) Find the solution of the initial value problem $y' + 2y = te^{-2t}$, $y(1) = 0$.

$$IF = \exp\left(\int 2\,dt\right) = e^{2t}$$

$$\left(y e^{2t}\right)' = t$$

$$y e^{2t} = \frac{1}{2}t^2 + C \qquad y(1)=0 \Downarrow$$

$$0 \cdot e^2 = \frac{1}{2} 1^2 + C$$

$$-\tfrac{1}{2} = C$$

Answer: $y = \frac{1}{2}t^2 e^{-2t} - \frac{1}{2}e^{-2t}$

(II) Find an integrating factor μ and solve implicitly $y\,dx + (2xy - e^{-2y})\,dy = 0$.

$$1 \neq 2y$$

$$\mu_y y + \mu \cdot 1 = \mu_x (2xy - e^{-2y}) + \mu \cdot 2y$$

$$\mu = \mu(y) \Rightarrow \mu_x = 0 \ \& \ \mu' = \mu_y$$

$$\mu' y + \mu = 0 + \mu \cdot 2y \Rightarrow y\mu' + (1-2y)\mu = 0$$

$$\Rightarrow \left(y e^{-2y} \mu\right)' = 0$$

$$\mu = C \frac{1}{y} e^{2y}$$

$$e^{2y}\,dx + \left(2xe^{2y} - \tfrac{1}{y}\right) dy = 0$$

$$\Psi_x\,dx + \Psi_y\,dy$$

$$\Psi = xe^{2y} + H(y)$$

$$\Psi_y = 2xe^{2y} + H'(y) \to H'(y) = -\tfrac{1}{y}$$

Answer: $\mu(y) = \frac{1}{y}e^{2y}$ solution: $xe^{2y} - \log|y| = c$

(III) State the existence and uniqueness theorem for the ~~general~~ linear second order initial value problem. [Make a drawing if it helps.]

S'pose that IVP

① $y'' + py' + qy = g$

② $p, q, \& g$ are continuous on some interval (a,b)

③ $y(t_0) = y_0$ & $y'(t_0) = y_0'$ and $t_0 \in (a,b)$

Then there exists a unique sol'n to the IVP over the interval (a,b).

(IV) The differential equation $x^2y'' + 3xy' + y = 0$ has for $x > 0$ the solution $y_1(x) = x^{-1}$. Find a second solution by the method of reduction of order.

$$1\left(y_2 = vx^{-1}\right)$$
$$3x\left(y_2' = v'x^{-1} - vx^{-2}\right)$$
$$+x^2\left(y_2'' = v''x^{-1} - 2v'x^{-2} + 2vx^{-3}\right)$$

$$0 = xv'' - 2v' + 3v'$$

$$0 = v'' + \frac{1}{x}v' \qquad (x>0)$$

$$0 = (v'x)' \qquad IF = \exp\left(\int \frac{1}{x}\,dx\right) = x$$

$$C = v'x$$

$$C/x = v'$$

$$C\log x + D = v \qquad (x>0)$$

$$y_2 = (C\log x + D)(1/x)$$
$$= C\frac{\log x}{x} + D\frac{1}{x}$$

Answer: $\tilde{y}_2(x) = \frac{\log x}{x}$

(V) Use the method of Variation of Parameters to find a particular solution $Y(t)$ of the equation $ty'' - (1+t)y' + y = t^2e^{2t}$, $t > 0$, given that $y_1(t) = e^t$ and $y_2(t) = 1+t$ form a fundamental system for the homogeneous equation.

$$y_H = C_1 y_1 + C_2 y_2$$

$$Y = y_I = u_1 y_1 + u_2 y_2$$

$$y'' - \left(\frac{1+t}{t}\right) y' + \frac{1}{t} y = te^{2t}$$

$$u_1 = -\int \frac{RHS \cdot y_2}{W(y_1, y_2)} \, dt$$

$$u_2 = +\int \frac{RHS \cdot y_1}{W(y_1, y_2)} \, dt$$

$$W(y_1, y_2) = \begin{vmatrix} e^t & 1+t \\ e^t & 1 \end{vmatrix}$$

$$= e^t - (1+t)e^t$$

$$= -te^t$$

$$u_2 = +\int \frac{te^{+2t} \cdot e^t}{-te^t} \, dt$$

$$= -\int e^{+2t} \, dt = -\frac{1}{2} e^{+2t}$$

$$u_1 = -\int \frac{te^{+2t} \cdot (1+t)}{-te^t} \, dt$$

$$= +\int (1+t) e^{\cancel{2}t} \, dt$$

$$= te^t$$

Answer: $Y(t) = te^{2t} + (1+t)\left(-\frac{1}{2}e^{2t}\right)$