I) Find the Laplace–inverse of $F(s) = \dfrac{e^{-10s}}{s^3 - s^2 - 6s} = e^{-10s} \cdot \dfrac{1}{s(s-3)(s+2)}$

$$\frac{1}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$$

$@\ s=0: \ \dfrac{1}{(0-3)(0+2)} = A = -\dfrac{1}{6}$

$@\ s=3: \ \dfrac{1}{3(3+2)} = B = \dfrac{1}{15}$

$@\ s=-2 \ \ \dfrac{1}{(-2)(-2-3)} = C = \dfrac{1}{10}$

$$F(s) = e^{-10s}\Big(\underbrace{-\frac{1}{6}\cdot\frac{1}{s} + \frac{1}{15}\cdot\frac{1}{s-3} + \frac{1}{10}\cdot\frac{1}{s+2}}_{}\Big)$$

$\mathcal{L}^{-1} \downarrow$ alone (w/o e^{-10s})

$$-\frac{1}{6} + \frac{1}{15}e^{3t} + \frac{1}{10}e^{-2t}$$

Answer: $f(t) =$ $-\dfrac{1}{6} + \dfrac{1}{15}e^{3(t-10)} + \dfrac{1}{10}e^{-2(t-10)}$

II) **Use Laplace transforms!** to solve the IVP $y'' - 6y' + 34y = 0$, $y(0) = 0$, $y'(0) = 1$.

$Y = \mathcal{L}(y)$

$(s^2Y - s\cdot 0 - 1) - 6(sY - 0) + 34Y = 0$

$(s^2 - 6s + 34)Y = 1$

$Y = \dfrac{1}{(s-3)^2 + 5^2} \cdot \dfrac{5}{5}$

$s^2 - 6s + 34 =$

$s^2 - 6s + 9 - 9 + 34 =$

$(s-3)^2 + 25$

Answer: $y(t) =$ $\dfrac{1}{5}e^{3t}\sin(5t)$

I) Find the Laplace–inverse of $F(s) = \frac{6}{s^2 - 10s + 34} = \frac{6}{(s-5)^2+3^2} = 2 \cdot \frac{3}{(s-5)^2+3^2}$

$$s^2 - 10s + 34 = s^2 - 10s + 25 - 25 + 34$$

$$= (s-5)^2 + 9$$

Answer: $f(t) = 2e^{5t} \sin(3t)$

II) **Use Laplace transforms!** to solve the IVP $y'' - y' - 2y = u_6(t)$, $y(0) = 0$, $y'(0) = 0$.

$$\mathcal{L}(y) = Y : \ (s^2Y - 0s - 0) - (sY - 0) - 2Y = e^{-6s}/s$$

$$(s^2 - s - 2)Y = e^{-6s}/s$$

$$Y = e^{-6s} \cdot \frac{1}{s(s-2)(s+1)}$$

$$Y = e^{-6s}\left(\underbrace{-\frac{1}{2} \cdot \frac{1}{s} + \frac{1}{6}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s+1}}\right)$$

$\mathcal{L}^{-1} \downarrow$ alone (w/o e^{-6s})

$$-\frac{1}{2} + \frac{1}{6}e^{2t} + \frac{1}{3}e^{-t}$$

Answer: $y(t) = -\frac{1}{2} + \frac{1}{6}e^{2(t-6)} + \frac{1}{3}e^{-(t-6)}$

III) Complete the following definitions concerning the SOLODE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (*$$

with polynomial coefficients P, Q, R:

(a) x_0 is an ordinary point for $(*)$ if $P(x_0) \neq 0$;

(b) x_0 is a singular point for $(*)$ if $P(x_0) = 0$;

(c) x_0 is a regular singular point for $(*)$ if x_0 is a singular pt and

$$\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} \quad \text{exist}$$

IV) Find a fundamental system of solutions of $x^2y'' - 3xy' + 4y = 0$.

$$r(r-1) - 3r + 4 = 0$$

$$(r-2)^2 = 0$$

$$\Rightarrow y_1 = x^2 \; \& \; y_2 = x^2 \log x$$

(that is, the gen'l sol'n is $y = C_1 x^2 + C_2 x^2 \log x$)

IV) Find the gen'l sol'n of $x^2y'' + 3xy' + 5y = 0$.

$$r(r-1) + 3r + 5 = 0$$

$$r^2 + 2r + 5 = 0$$

$$(r+1)^2 + 4 = 0$$

$$r = -1 \pm 2i$$

$$y = C_1 x^{-1} \cos(2\log x) + C_2 x^{-1} \sin(2\log x)$$

(That is, a fund. sol'n system is

$$y_1 = x^{-1}\cos(2\log x)$$

$$\& \; y_2 = x^{-1}\sin(2\log x).$$

)

V) Solve the equation $y'' + 3x^2 y = 0$ by means of a power series about $x_0 = 0$; find the recurrence relation and the first three **non–zero** terms in each of two linearly independent solutions.

$$\left.\begin{matrix} 1 \neq 0 \\ 0 \,\&\, 3x^2 \text{ are polys} \end{matrix}\right\} \Rightarrow x = 0 \text{ is an ordinary pt}$$

$$3x^2 \left(y = \sum_0^\infty a_n x^n \right)$$

$$0 \left(y' = \sum_1^\infty n a_n x^{n-1} \right)$$

$$1 \left(y'' = \sum_2^\infty n(n-1) a_n x^{n-2} \right)$$

$$0 = \sum_2^\infty n(n-1) a_n x^{n-2} + \sum_0^\infty 3a_n x^{n+2}$$

$$0 = \sum_{-2}^\infty (n+4)(n+3) a_{n+4} x^{n+2} + \sum_0^\infty 3a_n x^{n+2}$$

$$0 = 2a_2 + 6a_3 x + \sum_0^\infty \left((n+4)(n+3) a_{n+4} + 3a_n \right) x^{n+2}$$

$$\Rightarrow 2a_2 = 0 \; ; 6a_3 = 0 \; ; \quad (n+4)(n+3) a_{n+4} + 3a_n = 0 \quad n \geq 0$$

a_0, a_1 are free

$$a_2 = a_3 = 0 \Rightarrow a_6 = a_7 = 0 \Rightarrow a_{10} = a_{11} = 0 \Rightarrow \dots$$

$$a_4 = -\frac{1}{4} a_0 \; ; \quad a_5 = -\frac{3}{20} a_1 \Rightarrow a_8 = \left(-\frac{3}{56}\right)\left(-\frac{1}{4} a_0\right) = \frac{3}{224} a_0$$

$$a_9 = \left(-\frac{1}{24}\right)\left(-\frac{3}{20} a_1\right) = \frac{1}{160} a_1$$

$$y = \underset{\times}{a_0} + \underset{\times}{a_1 x} + \underset{0}{a_2 x^2} + \underset{0}{a_3 x^3} + \underset{\times}{a_4 x^4} + \underset{\times}{a_5 x^5} + \underset{0}{a_6 x^6} + \underset{0}{a_7 x^7} + \underset{\times}{a_8 x^8} + \underset{\times}{a_9 x^9} + \dots$$

$$y = a_0 + a_1 x - \frac{1}{4} a_0 x^4 - \frac{3}{20} a_1 x^5 + \frac{3}{224} a_0 x^8 + \frac{1}{160} a_1 x^9 + \dots$$

Answer: recurrence relation: $a_2 = 0 = a_3 \; ; \quad a_{n+4} = -3a_n / (n+4)(n+3) \quad n \geq 0$

$y_1 = 1 - \frac{1}{4} x^4 + \frac{3}{224} x^8 + \dots$, $y_2 = x - \frac{3}{20} x^5 + \frac{1}{160} x^9 + \dots$

VI) Show that the differential equation $2xy'' + y' + xy = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation, the exponents at the singularity, and the recurrence relation.

$$x^2 y'' + \underbrace{\left(1 \cdot \frac{x}{2x}\right)}_{\downarrow \text{ as } x \to 0} x y' + \underbrace{\left(x \cdot \frac{x^2}{2x}\right)}_{\downarrow \text{ as } x \to 0} y = 0$$

$$x^2 Y'' + \frac{1}{2} x Y' + 0 \; Y = 0$$

(yes, $x=0$ is a regular singular pt)

$$r(r-1) + \frac{1}{2} r + 0 = 0$$

$$r^2 - \frac{1}{2} r = 0 \implies r = 0, 1/2$$

$$Y = C_1 x^0 + C_2 x^{1/2}$$

$$y = \left(\sum_0^\infty a_n x^n\right) x^0 + \left(\sum_0^\infty b_n x^n\right) x^{1/2}$$

Find both recurrence rel'ns at the same time:

$$x \left(y = \sum_0^\infty c_n x^{n+r} \right)$$ with r as above & $c=a$ for $r=0$, $c=b$ for $r=1/2$

$$1 \left(y' = \sum_0^\infty (n+r) c_n x^{n+r-1} \right)$$

$$2x \left(y'' = \sum_0^\infty (n+r)(n+r-1) c_n x^{n+r-2} \right)$$

$$0 = \sum_0^\infty 2(n+r)(n+r-1) c_n x^{n+r-1} + \sum_0^\infty (n+r) c_n x^{n+r-1} + \underbrace{\sum_0^\infty c_n x^{n+r+1}}$$

$$\begin{matrix} 0 \\ \| \end{matrix}$$

$$2r(r-1)c_0 x^{r-1} + 2r(1+r)c_1 x^r + \sum_2^\infty 2(n+r)(n+r-1)c_n x^{n+r-1} + r c_0 x^{r-1} + (1+r) c_1 x^r + \sum_2^\infty (n+r) c_n x^{n+r-1} + \sum_2^\infty c_{n-2} x^{n+r-1}$$

$$0 = \underbrace{\left(2r(r-1) + r\right)}_{\substack{\| \\ 0 \text{ for our roots}}} c_0 x^{r-1} + \underbrace{\left(2(r+1)r + (1+r)\right)}_{\substack{\neq \\ 0 \text{ for our roots so } c_1 = 0}} c_1 x^r + \sum_2^\infty \left[2(n+r)(n+r-1) c_n + (n+r) c_n + c_{n-2} \right] x^{n+r-1}$$

Answer: indicial equation: $r(r-1) + \frac{1}{2} r + 0 = 0$ exponents: $0, 1/2$

recurrence relation: $c_1 = 0$ & $c_n = -\dfrac{c_{n-2}}{2(n+r)(n+r-1) + (n+r)}$ for $n \geq 2$