I) (a) Define the Laplace transform of a function $f(t)$.

$$\mathcal{L}\{f(t)\}(s) = F(s) \stackrel{\text{def}}{=} \int_0^\infty f(t)\, \exp(-st)\, dt$$

(b) For which kind of differential equation problem is the Laplace transform a useful tool?

linear ODEs w/ constant coefficients

(c) How does it work on such a differential equation?

Laplace transforms change a differential equation into an easy-to-solve algebraic equation.

II) Write the solution of $y'' + 4y = u_\pi(t), y(0) = 0, y'(0) = 0$ as a convolution. **Write the convolution explicitly as an integral.**

$$\mathcal{L}(y) = Y$$

$$(s^2 Y - s \cdot 0 - 0) + 4Y = e^{-\pi s}/s$$

$$(s^2 + 4)Y = e^{-\pi s}/s$$

$$Y = \frac{e^{-\pi s}}{s} \cdot \left(\frac{1}{s^2+4} \cdot \frac{2}{2}\right)$$

$$y = u_\pi(t) * \frac{1}{2} \sin(2t)$$

Answer: $y = \int_0^t u_\pi(\tau) \cdot \frac{1}{2} \sin(2(t-\tau))\, d\tau$

III) Find the **explicit!** solution of $y'' + 4y = \delta(t-\pi), y(0) = 1, y'(0) = 0$.

$$\mathcal{L}(y) = Y$$

$$(s^2 Y - s \cdot 1 - 0) + 4Y = e^{-\pi s}$$

$$(s^2 + 4)Y - s = e^{-\pi s}$$

$$Y = e^{-\pi s} \cdot \frac{1}{s^2+4} + \frac{s}{s^2+4}$$

$$y = u_\pi(t) \cdot \frac{1}{2} \sin(2(t-\pi)) + \cos(2t)$$

IV) Suppose $y' = f(t,y)$ is to be solved numerically on the interval $[3,8]$ with initial condition $y(3) = 1$ and with step size h.

a) State how the local and global errors depend on h and how many evaluations of f are needed, for the Euler, Improved Euler, and Runge–Kutta methods, respectively.

Method	Local Error	Global Error	**Total** Nr of Evals of f
Euler	$e \sim h^2$	$E \sim h$	$\# = N$
Improved Euler	$e \sim h^3$	$E \sim h^2$	$\# = 2N$
Runge–Kutta	$e \sim h^5$	$E \sim h^4$	$\# = 4N$

$$N = \frac{8-3}{h}$$

b) Assume now specifically that $f(t,y) = t + y$. Approximating the solution using the Heun method, also called the improved Euler method, with step size $h = 0.2$, compute the point (t_1, y_1) you arrive at after the first step.

$$t_0 = 3 \quad y_0 = 1$$

$$k_0 = f(t_0, y_0) = 3 + 1 = 4$$

$$\tilde{Y} = y_0 + h k_0 = 1 + 0.2 \cdot 4 = 1.8$$

$$k_1 = f(t_1, \tilde{Y}) = 3.2 + 1.8 = 5$$

$$t_1 = 3.2 \quad y_1 = 1 + \left(\tfrac{1}{2}(4+5)\right) \cdot 0.2 = 1.9$$

$$\left(y_1 = y_0 + \left(\tfrac{1}{2}(k_0 + k_1)\right) h \right)$$

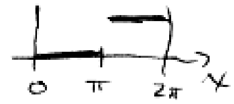
Answer: $t_1 = 3.2$ $\qquad y_1 = 1.9$

V) Solve the heat conduction problem $u_t = 3u_{xx}$ in an insulated rod of length 2π whose ends are maintained at 0° Celsius at all times and whose initial temperature $u(x,0)$ is given by $u(x,0) = f(x) \quad \forall x \in [0, 2\pi]$, where

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le \pi \\ 1 & \text{for } \pi < x \le 2\pi. \end{cases}$$

$$u(x,t) = \sum_{1}^{\infty} c_n \exp\left(-3\left(\frac{n\pi}{2\pi}\right)^2 t\right) \sin\left(\frac{n\pi}{2\pi} x\right)$$

$$\Rightarrow \quad u(x,0) = \sum_{1}^{\infty} c_n \sin\left(\frac{nx}{2}\right) =$$



$$\Rightarrow \quad c_n = \frac{2}{2\pi} \int_0^{2\pi} f(x) \sin\left(\frac{nx}{2}\right) dx$$

$$= \frac{1}{\pi} \int_{\pi}^{2\pi} \sin\left(\frac{nx}{2}\right) dx$$

$$= \frac{1}{\pi} \left[-\frac{2}{n} \cos\left(\frac{nx}{2}\right) \right]_{\pi}^{2\pi}$$

$$= -\frac{2}{n\pi} \left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right]$$

Answer: $u(x,t) = \sum_{1}^{\infty} -\frac{2}{n\pi} \left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right] \exp\left(-3\left(\frac{n}{2}\right)^2 t\right) \sin\left(\frac{nx}{2}\right)$

I) (a) Define the Laplace transform of a function $f(t)$.

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$$\mathcal{L}(y) = Y$$

$$(s^2Y - s\cdot 0 - 0) - 4Y = e^{-s}/s$$

$$(s^2 - 4)Y = e^{-s}/s$$

$$Y = \frac{e^{-s}}{s}\cdot\frac{1}{s^2-4}$$

$$y = u_1(t) * \frac{1}{2}\sinh(2t)$$

Answer: $y = \int_0^t u_1(\tau)\frac{1}{2}\sinh(2(t-\tau))\,d\tau$

III) Find the **explicit!** solution of $y'' - 4y = \delta(t-1), y(0) = 1, y'(0) = 0$.

$$\mathcal{L}(y) = Y$$

$$(s^2 Y - s \cdot 1 - 0) - 4Y = e^{-s}$$

$$(s^2 - 4)Y - s = e^{-s}$$

$$Y = e^{-s} \cdot \frac{1}{s^2-4} + \frac{s}{s^2-4}$$

$$y = u_1(t) \cdot \frac{1}{2} \sinh(2(t-1)) + \cosh(2t)$$

IV) Suppose $y' = f(t, y)$ is to be solved numerically on the interval $[2, 6]$ with initial condition $y(2) = 1$ and with step size h.

a) State how the local and global errors depend on h and how many evaluations of f are needed, for the Euler, Improved Euler, and Runge–Kutta methods, respectively.

Method	Local Error	Global Error	**Total** Nr of Evals of f
Euler	$e \sim h^2$	$E \sim h$	$\# = N$
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Runge–Kutta	$e \sim h^5$	$E \sim h^4$	$\# = 4N$

$N = \frac{6-2}{h}$

b) Assume now specifically that $f(t, y) = t + y$. Approximating the solution using the Heun method, also called the improved Euler method, with step size $h = 0.2$, compute the point (t_1, y_1) you arrive at after the first step.

$t_0 = 2 \qquad y_0 = 1$

$$k_0 = f(t_0, y_0) = 2 + 1 = 3$$

$$\tilde{y} = y_0 + h k_0 = 1 + 0.2 \cdot 3 = 1.6$$

$$k_1 = f(t_1, \tilde{y}) = 2.2 + 1.6 = 3.8$$

$t_1 = 2.2 \qquad y_1 = y_0 + \left(\frac{1}{2}(k_0 + k_1)\right)h$

$$= 1 + \left(\frac{1}{2}(3 + 3.8)\right)(0.2) = 1.68$$

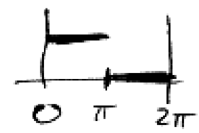
Answer: $t_1 =$ 2.2 $\qquad y_1 =$ 1.68

V) Solve the heat conduction problem $u_t = 2u_{xx}$ in an insulated rod of length 2π whose ends are maintained at $0°$ Celsius at all times and whose initial temperature $u(x,0)$ is given by $u(x,0) = f(x) \quad \forall\, x \in [0, 2\pi]$, where

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } \pi < x \le 2\pi. \end{cases}$$

$$u(x,t) = \sum_{1}^{\infty} c_n \exp\left(-2\left(\frac{n\pi}{2\pi}\right)^2 t\right) \sin\left(\frac{n\pi}{2\pi} x\right)$$

$$\Rightarrow \quad u(x,0) = \sum_{1}^{\infty} c_n \sin\left(\frac{nx}{2}\right) =$$



$$\Rightarrow \quad c_n = \frac{2}{2\pi}\int_0^{2\pi} f(x) \sin\left(\frac{nx}{2}\right) dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin\left(\frac{nx}{2}\right) dx$$

$$= \frac{1}{\pi}\left[-\frac{2}{n}\cos\left(\frac{nx}{2}\right)\right]_0^{\pi}$$

$$= \frac{-2}{\pi n}\left[\cos\left(\frac{n\pi}{2}\right) - \cos(0)\right]$$

Answer: $u(x,t) = \sum_{1}^{\infty} -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] \exp\left(-2\left(\frac{n}{2}\right)^2 t\right) \sin\left(\frac{nx}{2}\right)$