What's a function? A function is a rule that gives you one output/answer for a given input. For instance, $y(t) = t^2 + 6t + 5$ is a rule describing a function taking in a real number & spitting out its square plus 6 times itself plus 5.

A slightly more complicated example is:

$$y(t) = |t+5| - 2 = \begin{cases} t+3 & \text{if } t \geq -5 \\ -t-7 & \text{if } t < -5 \end{cases}$$

This rule still defines a function. However,

$$y(t) = \begin{cases} 2t+5 & \text{if } t < 0 \\ -6t-9 & \text{if } -1 < t < 6 \\ t^2+3 & \text{if } 2 < t \end{cases}$$

doesn't work as a rule for a function (what's $y(3)$ for instance? Is it 12 or -27?). A mathematician might say this is not a well-defined function. The rule

$$y(t) = \int_0^t \frac{ds}{1+s^5}$$

also is not well-defined when $t < -1$.

Look at a simpler example:

$$y(t) = \int_0^t (2s+3)\,ds \qquad \underline{\text{vs}} \qquad y(t) = \int_6^t \frac{ds}{s^2}$$

On the left, applying the Fundamental Theorem of Calculus, we get:

$$\int_0^t (2s+3)\,ds = s^2 + 3s\Big]_0^t = t^2 + 3t$$

This ~~is~~ might not be OK because the FTC also says I can choose any anti-deriv of $2s+3$:

$$\int_0^t (2s+3)\,ds = s^2 + 2s + C\Big]_0^t = t^2 + 3t$$

But things work out all-right 'cuz $C - C = 0$.

However

$$\int \frac{ds}{s^2} = -\frac{1}{s} + C$$

does not describe all anti-derivs of $\frac{1}{s^2}$.

For instance, if $f(s) = \frac{1}{s^2}$ and

$$F(s) = \begin{cases} -\frac{1}{s} + 3 & \text{if } s > 0 \\ -\frac{1}{s} + 7 & \text{if } s < 0 \end{cases}$$

then $F'(s) = f(s)$ over the whole domain so we say F is an anti-deriv of f.

Going back to:

$$y(t) = \int_6^t \frac{ds}{s^2}$$

According to the FTC:

$$y(t) = \int_6^t \frac{ds}{s^2} = -\frac{1}{s} + C \Big]_6^t = -\frac{1}{t} + \frac{1}{6}$$

no matter what "C" is. However,

$$\int_6^t \frac{ds}{s^2} = \left\{ \begin{matrix} -\frac{1}{s} + 3 & \text{if } s > 0 \\ -\frac{1}{s} + 7 & \text{if } s < 0 \end{matrix} \right\} \Bigg]_6^t = \left\{ \begin{matrix} -\frac{1}{t} + \frac{1}{6} & \text{if } t > 0 \\ -\frac{1}{t} + \frac{1}{6} + 4 & \text{if } t < 0 \end{matrix} \right\}$$

These two "definitions" agree if $t > 0$, but they don't if $t < 0$. This isn't too surprising since the FTC requires <u>continuity</u> of the integrand (note we haven't talked about $t=0$).

You might say, "hey, they agree for $t > 0$ because $6 > 0$. Why not just use the same "C" (const. of integration) for $t > 0$ & $t < 0$?" And that seems like a reasonable thing to do, and it comes up often enough that mathematicians have a name for it.

If we pick the same "C" no matter the input, then we're looking at the (Cauchy) principal value of the anti-deriv. (at least in this case) In that case,

$$\int_6^t \frac{ds}{s^2} = -\frac{1}{s} + C\Big]_6^t = -\frac{1}{t} + \frac{1}{6} \quad \text{(for all } t \neq 0)$$

you might write $PV\int\left(\frac{1}{s^2}\right)ds$ to be explicit.

Back to the quiz: since (with $\gamma = \frac{1+\sqrt{5}}{2}$ the golden ratio)

$$\frac{1}{1+s^5} = \frac{1}{5}\left[\frac{1}{1+s} + \frac{-\gamma s + 2}{s^2 - \gamma s + 1} + \frac{(1/\gamma)s + 2}{s^2 + (1/\gamma)s + 1}\right]$$

then

$$PV\int \frac{ds}{1+s^5} = \frac{1}{5}\left[\log|1+s| - \frac{\gamma}{2}\log\left(s^2 - \gamma s + 1\right) + 2\tan^{-1}\left(\frac{s - \gamma/2}{\sqrt{1-\gamma^2/4}}\right) + \frac{1}{2\gamma}\log\left(s^2 + (1/\gamma)s + 1\right) + 2\tan^{-1}\left(\frac{s + 1/2\gamma}{\sqrt{1 - 1/4\gamma^2}}\right)\right] + C$$

and

$$PV\int_0^t \frac{ds}{1+s^5} = \frac{1}{5}\left[\log|1+s| - \frac{\gamma}{2}\log(s^2 - \gamma s + 1) + 2\tan^{-1}\left(\frac{s - \gamma/2}{\sqrt{1-\gamma^2/4}}\right) + \frac{1}{2\gamma}\log\left(s^2 + (1/\gamma)s + 1\right) + 2\tan^{-1}\left(\frac{s + 1/2\gamma}{\sqrt{1 - 1/4\gamma^2}}\right) + 2\sin^{-1}\left(\left(\frac{1}{4}\right)\left(\sqrt{5+2\sqrt{5}} - \sqrt{5-2\sqrt{5}}\right)\right)\right]$$

OOPS!: The RHS above should have all "t"s not "s"s!

(NB: $\lim_{t\to\infty} y = \frac{2}{5}\left(\pi + \sin^{-1}\left(\left(\frac{1}{4}\right)\left(\sqrt{5+2\sqrt{5}} - \sqrt{5-2\sqrt{5}}\right)\right)\right)$ a constant, as expected)

I think I may have introduced a typo above; I'm not sure.

So why choose the principal value (over all other choices – a lot of them)? There's a connection with the idea of analytic continuation in complex analysis (the calculus of complex numbers) — it makes for elegant analytics. (This much should be obvious to you — you get to pick the same "C" on both sides of the pole/vertical asymptote.)

This is sometimes useful in practice. Sometimes not.

What does this have to do with DE? ~~[illegible]~~ The problem above arose in starting at a point and integrating ~~[illegible]~~ over a discontinuity. In solving DEs, we start at a point (the IC given from a physical model) and integrate to the point we've interested in. If we integrate past a discontinuity, we have to ask the model for more information (more than the DE plus IC) to decide what happens past the discontinuity (what new "C" to pick). Maybe the model requires analytic continuation (and the principal value) or maybe it doesn't.