Find the terms up to 4th order in the series centered at $x=2$ for the solution of the IVP:

$$(1+x^3)y'' + 4xy' + y = 0$$

$$y(2) = 1 \quad \& \quad y'(2) = 0$$

What is the radius of convergence of this series?

$$y = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \dots$$

$$\Rightarrow \quad y(2) = a_0, \; y'(2) = a_1, \; y''(2) = 2a_2$$

$$y'''(2) = 6a_3, \quad y^{(4)}(2) = 24a_4$$

IC: $1 = y(2) = a_0$

$0 = y'(2) = a_1$

DE: $\cancel{y}\,(1+x^3)\,y''(x) + 4x\,y'(x) + y(x) = 0$

$$\Rightarrow (1+2^3)\,y''(2) + 4\cdot 2\cdot y'(2) + y(2) = 0$$

$$\Rightarrow 18a_2 + 8a_1 + a_0 = 0 \quad \Rightarrow 18a_2 + 1 = 0$$

$$\Rightarrow a_2 = -\frac{1}{18}$$

$$(1+x^3)\,y'' + 4xy' + y = 0$$

$$(1+x^3)\,y''' + 3x^2y'' + 4xy'' + 4y' + y' = 0$$

$$9\cdot 6a_3 + \cancel{8}12\cdot 2a_2 + 8\cdot 2a_2 + 5\cdot a_1 = 0$$

$$54a_3 - \frac{4}{3} - \frac{8}{9} = 0 \quad \Rightarrow \quad a_3 = \frac{10}{243}$$

$$(1+x^3)\,y^{(4)} + 3x^2y''' + (3x^2+4x)\,y''' + (6x+4)\,y'' + 5y'' = 0$$

$$9\cdot 24a_4 + 12\cdot 6a_3 + 20\cdot 6a_3 + 16\cdot 2a_2 + 5\cdot 2a_2 = 0$$

$$9\cdot 24a_4 + \underbrace{\frac{80}{27} + \frac{400}{81}}_{\frac{640}{81}} \underbrace{- \frac{32}{18} - \frac{10}{18}}_{-\frac{7}{3}} = 0$$

$$9\cdot 24a_4 = -\frac{640 - 27\cdot 7}{81} = -\frac{451}{81}$$

$$a_4 = -\frac{451}{17496}$$

$$y = 1 - \frac{1}{18}(x-2)^2 + \frac{10}{243}(x-2)^3 - \frac{451}{17496}(x-2)^4 + \cdots$$

ROC = distance from the CTR of the series to the nearest SINGULAR point

$$1 + x^3 = 0 \quad \Rightarrow \quad x = -1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \text{or}\ \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$|2-(-1)| = 3 \qquad \left|2 - \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right| = \sqrt{3}$$

$$\Rightarrow \text{ROC} = \sqrt{3}$$