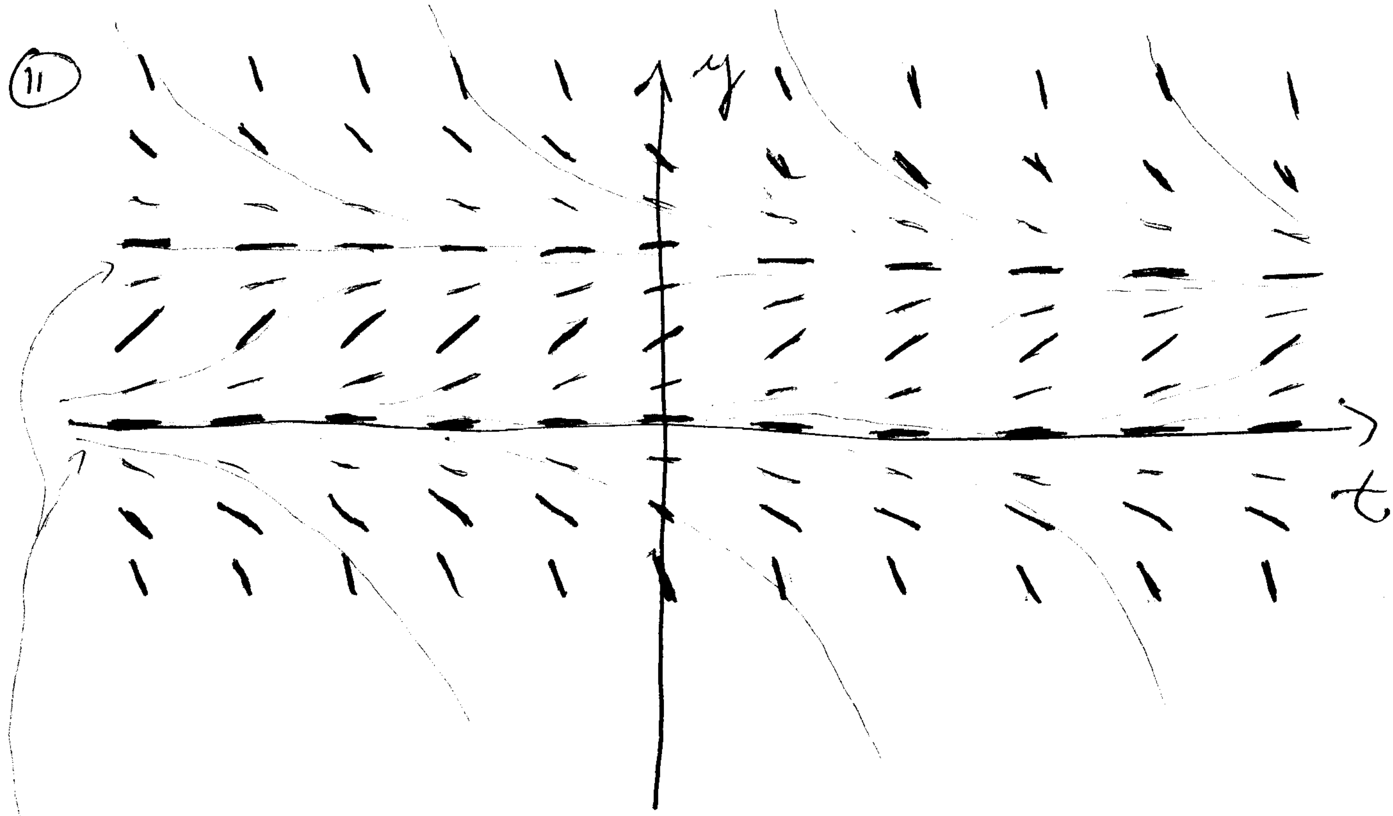
§ 1.1

(11)



$$y' = y(4-y)$$

$$y \equiv 0 \quad \text{or} \quad y \equiv 4 \iff y' \equiv 0$$

$$y > 4 \Rightarrow y' < 0$$

$$0 < y < 4 \Rightarrow y' > 0$$

$$y < 0 \Rightarrow y' < 0$$

$$\lim_{t \to \infty} y = 4 \quad \text{if} \quad y(0) > 0$$

$$\lim_{t \to \infty} y = -\infty \quad \text{if} \quad y(0) < 0$$

$$\lim_{t \to \infty} y = 0 \quad \text{if} \quad y(0) = 0$$

§1.3

② $(1+\boxed{y^2})\,y^{''} + t\,y' + y = e^t$

order = 2

non linear

④ $y^{1} + \boxed{t\,y^2} = 0$

order = 1

non linear

⑦ $y'' - y = 0$

$y_1 = e^t$

$y_1' = e^t$

$y_1'' = e^t$

$y_1'' - y_1 = e^t - e^t = 0$ ✓

$y_2 = \cosh t$

$y_2' = \sinh t$

$y_2'' = \cosh t$

$y_2'' - y_2 = \cosh t - \cosh t$

$= 0$ ✓

⑭ $y' - 2ty = 1$

$y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$

$y' = 2t\,e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}\cdot e^{-t^2} + 2te^{t^2}$

$-\ 2ty = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 2te^{t^2}$

$1 = e^{t^2}\cdot e^{-t^2} = e^0$ ✓

§2.1

(6)(c) $ty' + 2y = \sin t$ for $t > 0$

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

$$IF = \exp\left(\int \frac{2}{t}\,dt\right)$$

$$= \exp(2\log t)$$

$$= \exp(\log(t^2)) = t^2$$

$$\frac{d}{dt}(t^2 y) = \frac{\sin t}{t}\cdot t^2$$

$$t^2 y = \int t\sin t\,dt$$

$$t^2 y = -t\cos t + \sin t + C$$

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$$

$$\lim_{t\to\infty} y = 0 + 0 + 0 = 0$$

(irrespective of C)

(18) From 6(c) we know the gen'l sol'n to $ty' + 2y = \sin t$ is

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$$

so that

$$1 = y(\pi/2) = -0 + 1/(\pi/2)^2 + C/(\pi/2)^2$$

and

$$C = (\pi/2)^2 - 1$$

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{(\pi/2)^2 - 1}{t^2}$$

(30) $y' + ay = be^{-\lambda t}$

$IF = \exp\left(\int a\,dt\right) = e^{at}$ (so long as $a \neq 0$ which it is since $a > 0$ by assumption)

$$\frac{d}{dt}\left(e^{at} y\right) = be^{-\lambda t} \cdot e^{at}$$

$$e^{at} y = \int be^{(a-\lambda)t}\,dt$$

Case 1: $a = \lambda$

Then $a - \lambda = 0$ and $\int be^{(a-\lambda)t}\,dt = \int b\,dt = bt + C$

so that $e^{at} y = bt + C$

$$y = (bt + C)e^{-at}$$

since $\lim_{t\to\infty} e^{-at} = 0 = \lim_{t\to\infty} te^{-at}$ then $\lim_{t\to\infty} y = 0$

'cuz $a > 0$

(30) (cont)

Case 2: $a \neq \lambda$

Then $a-\lambda \neq 0$ and

$$\int b e^{(a-\lambda)t}\, dt = \frac{b}{a-\lambda} e^{(a-\lambda)t} + C$$

so that

$$e^{at} y = \frac{b}{a-\lambda} e^{(a-\lambda)t} + C$$

$$y = \frac{b}{a-\lambda} e^{-\lambda t} + Ce^{-at}$$

$$\lim_{t\to\infty} e^{-\lambda t} = 0 = \lim_{t\to\infty} e^{-at}$$

'cuz $\lambda > 0$ & $a > 0$

Thus $\lim_{t\to\infty} y = 0$

§2.2

(29) Case 1: $a \neq 0$

$$y' = \frac{ay+b}{cy+d} \implies \frac{cy+d}{ay+b}\, dy = dx \quad \text{or} \quad y = -b/a$$

(This assumes $cy+d \neq 0$)

Using long division on the first part:

$$\frac{cy+d}{ay+b} = \frac{c}{a} + \frac{d - bc/a}{ay+b}$$

(NB: The second part, $y = -b/a$, represents a sol'n.)

(29) (cont).

Then $\int \left(\frac{c}{a} + \frac{d - bc/a}{ay+b} \right) dy = \int dx$

so that $\frac{c}{a} y + (d - bc/a) \cdot \frac{1}{a} \log|ay+b| = x + E$

Case 2: $a = 0$ & $b \neq 0$

then $\left(\frac{c}{b} y + \frac{d}{b} \right) dy = dx$

and $\frac{c}{2b} y^2 + \frac{d}{b} y = x + E$

Case 3: $a = 0$ & $b = 0$

$y' = 0 \Rightarrow y = E$ (a constant)

To summarize

$(a \neq 0)$ and $\left[(y = -b/a) \text{ or } \left((y \neq -b/a) \text{ and } \left(\frac{c}{a} y + (d - bc/a) \cdot \frac{1}{a} \log|ay+b| = x+E \right) \right) \right]$

or

$(a = 0)$ and $(b \neq 0)$ and $\left(\frac{c}{2b} y^2 + \frac{d}{b} y = x + E \right)$

or

$(a = 0)$ and $(b = 0)$ and $(y = E)$

§2.4

(8) $y' = (1-t^2-y^2)^{1/2}$ so $f = (1-t^2-y^2)^{1/2}$

and $\frac{\partial f}{\partial y} = -y(1-t^2-y^2)^{-1/2}$

f is continous for $1-t^2-y^2 \geq 0$ (radicand ≥ 0)

$\frac{\partial f}{\partial y}$ is continous for $1-t^2-y^2 > 0$ (radicand ≥ 0 & denominator $\neq 0$)

The hypotheses of Thm 2.4.2 are satisfied whenever $1-t^2-y^2 > 0$ (if you really want a rectangle, note that $t^2+y^2<1$ describes the unit disk; pick $-\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt{2}}$ & $-\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}}$ for instance – there're lots of rectangles contained in the unit disk).

(28) $t^2y' + 2ty - y^3 = 0$ for $t > 0$

Either $y=0$ or let $v = y^{1-3} \left(= y^{-2}\right)$.

Then $v' = -2y^{-3}y'$. Divide the DE by y^3 (to save on some algebra):

$$t^2\frac{y'}{y^3} + 2t\frac{1}{y^2} - 1 = 0$$

$$t^2 \cdot \frac{v'}{-2} + 2tv - 1 = 0 \qquad \text{(This is linear)}$$

$$v' - \frac{4}{t}v = -\frac{2}{t^2}$$

(28) (cont)

$$v' - \frac{4}{t} v = -\frac{2}{t^2}$$

$$IF = \exp\left(\int -\frac{4}{t}\, dt \right) = \exp(-4 \log t)$$

$$= \exp\left(\log\left(t^{-4}\right)\right) = t^{-4}$$

$$\frac{d}{dt}\left(t^{-4} v\right) = -\frac{2}{t^2} \cdot t^{-4} \quad \left(= -2t^{-6}\right)$$

$$t^{-4} v = \frac{-2}{-5} t^{-5} + C$$

$$v = \frac{2}{5} t^{-1} + C t^4$$

$v = 1/y^2$ so

$$y = \pm \left(\frac{2}{5} t^{-1} + C t^4\right)^{-1/2}$$