§ 2.2

(36) $$(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$$

can be written as:

$$\left(1 + 3(y/x) + (y/x)^2\right) - y' = 0$$

by dividing by $x^2\,dx$ (assuming $x \neq 0$).

Substituting $y = xv$

& $y' = v + xv'$:

$$(1 + 3v + v^2) - (v + xv') = 0$$

$$1 + 2v + v^2 = xv'$$

Separating:

$$\frac{dx}{x} = \frac{dv}{1 + 2v + v^2} \quad \underline{\text{or}} \quad v = -1$$

Integrating: (NB: $1 + 2v + v^2 = (1+v)^2$)

$$C + \log|x| = -(1+v)^{-1}$$

so that

$$v = -1 - \frac{1}{C + \log|x|}$$

and

$$y = -x - \frac{x}{C + \log|x|} \quad \underline{\text{or}} \quad y = -x$$

§2.3

(14) (a) If $Q' = -rQ$, then $Q' + rQ = 0$.

An IF is e^{rt} so that $(Qe^{rt})' = 0$

and $Qe^{rt} = C$ or $Q = Ce^{-rt}$.

For C_{14} the half-life is $t_{1/2} = 5730$ yr.

The half-life has the property that:

$$Q(t + t_{1/2}) = \tfrac{1}{2} Q(t)$$

From what we know above:

$$Ce^{-r(t+t_{1/2})} = \frac{1}{2} Ce^{-rt}$$

Dividing by Ce^{-rt} gives:

$$e^{-rt_{1/2}} = \frac{1}{2}$$

$$-rt_{1/2} = -\log 2$$

$$r = \frac{\log 2}{t_{1/2}} = \frac{\log 2}{5730} \cdot (\text{yr}^{-1})$$

(b)

$$Q_0 = Q(0) = Ce^{-r \cdot 0} = C$$

so that $Q(t) = Q_0 e^{-rt}$

(c) S'pose A is the age of the remains. We're told $Q(A) = \frac{1}{5} Q(0)$ so that

$$Q_0 e^{-rA} = \frac{1}{5} Q_0 e^{-r0} \Rightarrow e^{-rA} = \frac{1}{5} \Rightarrow -rA = -\log 5$$

$$\Rightarrow A = \frac{\log 5}{r} = 5730 \left(\frac{\log 5}{\log 2}\right) \text{yr}.$$

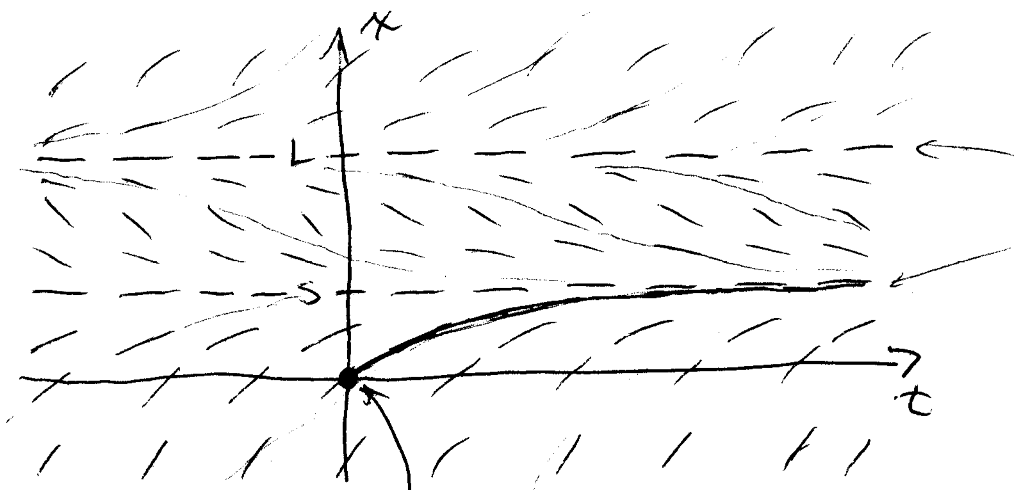
§2.5

(26) $x' = \alpha(p-x)(q-x)$ where $\alpha > 0$, $p \geq 0$, $q \geq 0$, and $p \neq q$.

initial concentrations (neg. conc. don't make sense)

(a) Let's draw a direction field for x & x'.

Let S be the smaller of p & q and
L be the larger of p & q.



we're told $x(0)=0$

From the direction field we see that $x = S$ is a stable equilibrium point. Since $0 \leq S$,
$\lim_{t\to\infty} x = S$ if $x(0)=0$ (in fact, if $x(0)<L$).

The DE is separable:

$$\frac{dx}{(p-x)(q-x)} = \alpha\, dt \quad \text{or} \quad x=p \quad \text{or} \quad x=q$$

$$\int\left(\frac{1}{q-p}\cdot\frac{1}{p-x} + \frac{1}{p-q}\cdot\frac{1}{q-x}\right)dx = \int \alpha\, dt$$

$$\frac{-1}{q-p}\log|p-x| + \frac{-1}{p-q}\log|q-x| = \alpha t + C$$

(2b) (a) (cont)

Since $x(0)=0$, we have

$$\frac{-1}{q-p}\log p + \frac{-1}{p-q}\log q = C$$

$$\shortparallel$$

$$\frac{-1}{p-q}\log\left(q/p\right)$$

We see from the direction field that $x<p$ & $x<q$ so $p-x>0$ & $q-x>0$ and we can drop the absolute value bars:

$$\frac{-1}{q-p}\log(p-x) + \frac{-1}{p-q}\log(q-x) = \alpha t - \frac{1}{p-q}\log\left(q/p\right)$$

$$-\log(p-x) + \log(q-x) = \left(\alpha(p-q)t - \log\left(q/p\right)\right)(-1)$$

$$\frac{q-x}{p-x} = \frac{q}{p}e^{-\alpha(p-q)t} = ☁$$

$$q-x = (p-x)\,☁$$

$$q-p\,☁ = x(1-☁)$$

$$x = \frac{q-p\,☁}{1-☁} = qp\,\frac{1-e^{-\alpha(p-q)t}}{p-qe^{-\alpha(p-q)t}}$$

(b) $p=q$ (and thus $L=S$) is simply the limiting case of p close to q as above.

The analytic form changes a bit to:

$$x = p\left(1-\frac{1}{1+\alpha pt}\right)$$

(NB: This is also the limiting case of (c).)

§2.6

(5) $y' = -\dfrac{ax+by}{bx+cy}$

In STD form for exact eqns, this is:

$$(ax+by) + (bx+cy)\,y' = 0$$

For this to be exact, we test:

$$\frac{\partial}{\partial y}(ax+by) \overset{?}{=} \frac{\partial}{\partial x}(bx+cy)$$

$$b = b \checkmark$$

$$\frac{\partial\psi}{\partial x} = ax+by \xrightarrow[\partial/\partial x]{\text{undo}} \psi = \tfrac{1}{2}ax^2 + bxy + C(y)$$

$$\frac{\partial\psi}{\partial y} = bx+cy \quad = \quad \frac{\partial\psi}{\partial y} = bx + C'(y) \quad (\partial/\partial y)$$

$$bx+cy = \cancel{}\,bx + C'(y)$$

$$cy = C'(y)$$

$$\tfrac{1}{2}cy^2 = C(y)$$

sol'n to DE

$$\psi = \boxed{\tfrac{1}{2}ax^2 + bxy + \tfrac{1}{2}cy^2 = D}$$

constant

⑥ $$\frac{dy}{dx} = -\frac{ax-by}{bx-cy}$$

$$(ax-by) + (bx-cy)y' = 0$$

$$\frac{\partial}{\partial y}(ax-by) \neq \frac{\partial}{\partial x}(bx-cy)$$

$$\underset{-b}{\parallel} \qquad\qquad \underset{b}{\parallel}$$

This is not exact.

⑬ $(2x-y)\,dx + (2y-x)\,dy = 0$ & $y(1)=3$

$$\frac{\partial}{\partial y}(2x-y) = -1 = \frac{\partial}{\partial x}(2y-x) \checkmark$$

$$\frac{\partial \psi}{\partial x} = 2x-y \xrightarrow[\partial/\partial x]{\text{undo}} \psi = x^2 - xy + C(y)$$

$$\frac{\partial \psi}{\partial y} = 2y-x \quad = \quad \frac{\partial \psi}{\partial y} = -x + C'(y)$$

$\partial/\partial y$

$$2y-x = -x + C'(y) \Rightarrow C'(y) = 2y$$

$$C(y) = y^2$$

sol'n to DE

$$\psi = \boxed{x^2 - xy + y^2 = D}$$

IC: $y=3$ when $x=1$ $\Rightarrow$ $1-3+9 = D$

$$\boxed{x^2 - xy + y^2 = 7}$$

sol'n to IVP

§2.9

(15) (a) $u_{n+1} = p\,u_n(1-u_n)$ with $p = 3.2$

	$u_0 = 0.2$ & 0.8	$u_0 = 0.4$ & 0.6
u_1	0.512	0.768
u_2	0.800	0.570
u_3	0.513	0.784
u_4	0.799	0.541
u_5	0.513	0.795
u_6	0.799	0.522
u_7	0.513	0.798
u_8	0.799	0.515
u_9	0.513	0.799
u_{10}	0.799	0.513
u_{11}	0.513	0.799
u_{12}	0.799	0.513
u_{13}		
u_{14}		
u_{15}		

They all eventually (big "n") oscillate between $u_n \cong 0.513$ & $u_{n+1} \cong 0.799$ (though they are out of phase). These agree w/ the calculated values of $\frac{21}{32}\left(1 \pm \sqrt{\frac{1}{21}}\right)$.