HW3

§3.1

(21) $y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r = 2$ or $-1$

$y = Ce^{2t} + De^{-t}$

$y' = 2Ce^{2t} - De^{-t}$

$$\left.\begin{array}{l} \alpha = y(0) = C + D \\ 2 = y'(0) = 2C - D \end{array}\right\} \Rightarrow \begin{array}{l} \alpha + 2 = 3C \\ C = \frac{\alpha+2}{3} \\ D = \frac{2\alpha-2}{3} \end{array}$$

$$y = \left(\frac{\alpha+2}{3}\right)e^{2t} + \left(\frac{2\alpha-2}{3}\right)e^{-t}$$

$$\lim_{t\to\infty} y = \left(\frac{\alpha+2}{3}\right)\underbrace{\lim_{t\to\infty} e^{2t}}_{+\infty} + \underbrace{\left(\frac{2\alpha-2}{3}\right)\lim_{t\to\infty} e^{-t}}_{0}$$

no matter what $\alpha$ is

$\Rightarrow$ we need $\frac{\alpha+2}{3} = 0$ so that $\lim_{t\to\infty} y = 0$.

$\Rightarrow \alpha = -2$

$y = -2e^{-t}$

(31) $2t^2 \frac{d}{dt}\left(\frac{dy}{dt}\right) + \left(\frac{dy}{dt}\right)^3 = 2t\frac{dy}{dt}$ for $t>0$

Substitute $v = \frac{dy}{dt}$ & eliminate $y$.

$$2t^2 \frac{dv}{dt} + v^3 = 2tv \qquad (\text{for } t>0)$$

This is a first order Bernoulli eqn.

Sub $w = v^{1-3} = v^{-2}$

$$\frac{dw}{dt} = -2v^{-3}\frac{dv}{dt}$$

$$\begin{cases} v=0 \\ \text{or} \\ 2t^2 \cdot \frac{1}{v^3}\frac{dv}{dt} + 1 = 2t\cdot\frac{1}{v^2} \end{cases} \qquad (\text{for } t>0)$$

$$2t^2\cdot\left(-\frac{1}{2}\frac{dw}{dt}\right) + 1 = 2tw$$

This is first order linear.

$$\frac{dw}{dt} + \frac{2}{t}w = \frac{1}{t^2} \qquad \text{IF} = \exp\left(\int \frac{2}{t}\,dt\right) = t^2$$

$$\frac{d}{dt}\left(t^2 w\right) = \frac{1}{t^2}\cdot t^2$$

$$t^2 w = t + C$$

$$v^{-2} = w = \frac{t+C}{t^2}$$

$$v = \pm\frac{t}{(t+C)^{1/2}} \quad \text{or} \quad v = 0$$

(31) (cont)

$$\frac{dy}{dt} = \pm \frac{t}{(t+C)^{1/2}} \qquad \text{or} \qquad \frac{dy}{dt} = 0$$

$$y = \pm \underbrace{\int \frac{t}{(t+C)^{1/2}}\, dt} \qquad \text{or} \qquad y = D$$

$u = t + C$
$du = dt$

$$\int \frac{u - C}{u^{1/2}}\, du = \int \left(u^{1/2} - C u^{-1/2}\right) du$$

$$= \frac{2}{3} u^{3/2} - 2u^{1/2} + D$$

$$y = \pm\left[\frac{2}{3}(t+C)^{3/2} - 2(t+C)^{1/2}\right] + D \quad \{\text{or } y = D\}$$

(39) $\frac{d}{dt}\left(\frac{dy}{dt}\right) + \left(\frac{dy}{dt}\right)^2 = 2e^{-y}$

Substitute $v = \frac{dy}{dt}$:

$$\frac{dv}{dt} + v^2 = 2e^{-y}$$

And eliminate $t$ (!) by noting $\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} \cdot v$:

$$v \frac{dv}{dy} + v^2 = 2e^{-y}$$ (We now treat $y$ as the independent variable.)

This is a first order Bernoulli equation. We substitute $w = v^2$ (to see this, divide the DE by $v$ to get (the almost linear but for $v^{-1}$) $\frac{dv}{dy} + v = (2e^{-y})\frac{1}{v}$.)

(39) (cont)

$$v \frac{dv}{dy} + v^2 = 2e^{-y}$$

$$w = v^2 \quad \text{and} \quad \frac{dw}{dy} = 2v \frac{dv}{dy}$$

$$\frac{1}{2} \frac{dw}{dy} + w = 2e^{-y}$$

$$\frac{dw}{dy} + 2w = 4e^{-y} \qquad IF = \exp\left(\int 2\,dy\right) = e^{2y}$$

$$\frac{d}{dy}\left(e^{2y} w\right) = 4e^{-y} e^{2y}$$

$$e^{2y} w = 4e^{y} + C$$

$$v^2 = w = \frac{4e^y + C}{e^{2y}}$$

$$v = \pm \frac{(4e^y + C)^{1/2}}{e^y} = \frac{dy}{dt} \qquad \text{(this is separable)}$$

$$\underbrace{\pm \int \frac{e^y}{(4e^y + C)^{1/2}}\, dy}_{} = \int dt$$

$u = 4e^y + C$
$du = 4e^y\, dy$

$$\int \frac{1/4\, du}{u^{1/2}} = \frac{1}{2} u^{1/2} + D = \frac{1}{2}(4e^y + C)^{1/2} + D$$

$$\frac{1}{2}(4e^y + C)^{1/2} + D = t$$

$$4e^y + C = 4(t - D)^2$$

$$e^y = (t - D)^2 - C/4$$

$$y = \log\left[(t - D)^2 - C/4\right]$$