§3.6

① $y'' - 2y' - 3y = 3e^{2t}$

$y = y_H + y_I$

$y_H: \quad y_H'' - 2y_H' - 3y_H = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$y_H = C_1 e^{3t} + C_2 e^{-t}$

$-3\,(y_I = Ae^{2t})$

$-2\,(y_I' = 2Ae^{2t})$

$+1\,(y_I'' = 4Ae^{2t})$

$3e^{2t} = -3Ae^{2t} \Rightarrow 3 = -3A$

$-1 = A$

$y = C_1 e^{3t} + C_2 e^{-t} - e^{2t}$

(19) (A) $$y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t)$$

$$y = y_H + y_I$$

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y_H: $$y_H'' + 3y_H' = 0$$

$$r^2 + 3r = 0$$

$$r = 0, -3$$

$$y_H = C_1 + C_2 e^{-3t}$$

$$y_I = (At^4 + Bt^3 + Ct^2 + Dt + E)t$$
$$+ (Ft^2 + Gt + H)e^{-3t} \cdot t$$
$$+ I\sin(3t) + J\cos(3t)$$

(30) S'pose $y = Y_1$ & $y = Y_2$ are sol'ns of the DE:

$$ay'' + by' + cy = g.$$

Then $\tilde{y} = Y_1 - Y_2$ is a sol'n of:

$$a\tilde{y}'' + b\tilde{y}' + c\tilde{y} = 0.$$

Further s'pose that $a > 0$, $b > 0$, & $c > 0$. We need to show that $\lim_{t\to\infty} \tilde{y} = 0$.

(30) (cont)

The eq'n for $\tilde{y}$ is linear & homog, and has const. coeff.

The char. eq'n for $\tilde{y}$ is:

$$ar^2 + br + c = 0$$

Whose roots are:

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ (NB: we assumed $a \neq 0$)

<u>Case 1:</u> $b^2-4ac > 0$

Since $a>0$ & $c>0$, $4ac>0$ and $b^2 > b^2-4ac$.
Thus $|b| > \sqrt{b^2-4ac}$ (this makes sense since $b^2-4ac>0$ and $\sqrt{b^2-4ac}$ is real). Since $b>0$, we have that $-b+\sqrt{b^2-4ac} < 0$ (and also more trivially $-b-\sqrt{b^2-4ac} < 0$). Since also $a>0$, both roots of the char. eq'n are negative. (Label the roots r_1 & r_2.)

Since $\tilde{y} = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, $\lim_{t\to\infty} \tilde{y} = 0$.

<u>Case 2:</u> $b^2-4ac = 0$

$$\tilde{y} = C_1 e^{(-b/2a)t} + C_2 t e^{(-b/2a)t}$$ (NB: $\frac{-b}{2a} < 0$ 'cuz $b>0$ & $a>0$)

Thus $\lim_{t\to\infty} \tilde{y} = 0$

(30) (cont).

Case 3: $b^2 - 4ac < 0$

Let $\mu = -\frac{b}{2a}$ & $\lambda = \frac{\sqrt{4ac-b^2}}{2a}$. Then the roots of the char. eqn are $\mu \pm i\lambda$.

Thus $\tilde{y} = C_1 e^{\mu t} \cos(\lambda t) + C_2 e^{\mu t} \sin(\lambda t)$

Since $\mu < 0$, $\lim_{t \to \infty} \tilde{y} = 0$.

§3.7

(5) $y'' + y = \tan t \qquad 0 < t < \pi/2$

$y = y_H + y_I$

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y_H: $\quad y_H'' + y_H = 0$

$r^2 + 1 = 0$

$$y_H = C_1 \underbrace{\cos t}_{y_1} + C_2 \underbrace{\sin t}_{y_2}$$

$$y_I = u_1 \cos t + u_2 \sin t$$

$$u_1 = -\int \frac{\tan t \cdot \sin t}{W(\cos t, \sin t)}\, dt$$

$$u_2 = +\int \frac{\tan t \cdot \cos t}{W(\cos t, \sin t)}\, dt$$

⑤ (cont)

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$u_1 = -\int \frac{\sin^2 t}{\cos t}\,dt = \int (\cos t - \sec t)\,dt$$

$$= \sin t - \log|\sec t + \tan t|$$

(NB: we've allowed to ignore the const. of integration here)

we can actually drop these since $\sec t + \tan t$ is positive on $0 < t < \pi/2$

$$u_2 = +\int \sin t\,dt = -\cos t$$

$$y_I = \big(\sin t + \log(\sec t + \tan t)\big)\cos t + (-\cos t)\sin t$$

$$= (\cos t)\log(\sec t + \tan t)$$

$$y = C_1 \cos t + C_2 \sin t + (\cos t)\log(\sec t + \tan t)$$

§3.8

⑭ We're told in the hint that $mx'' = -kx$ where m is the mass hanging from the spring, the spring has a force constant k, and x is the distance the spring has been stretched from its equilibrium point.

(14) (cont)

$$mx'' + kx = 0$$

$$mr^2 + k = 0$$

$$r = \pm i\sqrt{k/m}$$

$$x = C_1 \cos\left(\sqrt{k/m}\, t\right) + C_2 \sin\left(\sqrt{k/m}\, t\right)$$

Thus for any ICs, x oscillates w/ period

$$\frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{L}{g}}.$$

We're also told in the hint the equilibrium condition: $kL = mg$ where L is the length the spring is stretched at equilibrium, and g is the acceleration due to gravity.