1 2, 4, 6, 8, 18
2 2, 15a
3 11

§5.1

② The series $\sum_0^\infty \frac{nx^n}{2^n}$ converges if

$$\left.\begin{array}{l} \lim_{n\to\infty} \sqrt[n]{\left|\frac{nx^n}{2^n}\right|} < 1 \\ \quad \| \\ \left|\frac{x}{2}\right| \cdot \lim_{n\to\infty} \sqrt[n]{n} \end{array}\right\} \Rightarrow \begin{array}{l} |x|/2 < 1 \\ \text{or } |x| < 2 \end{array}$$

The radius of the interval of x's for which the series converges is 2.

④ The series $\sum_0^\infty 2^n x^n$ converges if

$$\left.\begin{array}{l} \lim_{n\to\infty} \left|\frac{2^{n+1} x^{n+1}}{2^n x^n}\right| < 1 \\ \quad \| \\ |2x| \cdot \lim_{n\to\infty} 1 \end{array}\right\} \Rightarrow \begin{array}{l} |2x| < 1 \\ \text{or } |x| < 1/2 \end{array}$$

ROC = 1/2

⑥ The series $\sum_1^\infty \frac{(x-x_0)^n}{n}$ converges if

$$\left.\begin{array}{l} \lim \sqrt[n]{\left|\frac{(x-x_0)^n}{n}\right|} < 1 \\ \quad \| \\ |x-x_0| \cdot \lim \sqrt[n]{1/n} \end{array}\right\} \Rightarrow \begin{array}{l} |x-x_0| < 1 \\ \text{ROC} = 1 \end{array}$$

(8) The series $\sum_{1}^{\infty} \frac{n!\,x^n}{n^n}$ converges if

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,x^{n+1} / (n+1)^{n+1}}{n!\,x^n / n^n} \right| < 1$$

$$\shortparallel$$

$$|x| \cdot \lim_{n\to\infty} \left\{ \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \right\} = |x| \cdot \lim_{n\to\infty} \left\{ (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} \right\}$$

$$\shortparallel$$

$$|x| \cdot \lim \left\{ \left(\frac{n}{n+1}\right)^n \right\} = |x| \cdot \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = |x|/e$$

$$\Rightarrow \quad |x|/e < 1 \quad \Rightarrow \quad |x| < e$$

ROC $= e$

(18)

$$y = \sum_{0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y' = \sum_{1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$y'' = \sum_{2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots$$

Also $y' = \sum_{0}^{\infty} (n+1) a_{n+1} x^n$ & $y'' = \sum_{0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

If $y = y''$ Then $a_0 = 2a_2$ & $a_1 = 6a_3$

or $a_2 = \frac{1}{2} a_0$ & $a_3 = \frac{1}{6} a_1$. Lastly for $n \geq 0$

$$a_n = (n+2)(n+1) a_{n+2} \quad \text{or} \quad a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

§5.2

② $y'' - xy' - y = 0$

$$-1\left(y = \sum_0^\infty a_n x^n\right)$$

$$-x\left(y' = \sum_1^\infty n a_n x^{n-1}\right)$$

$$+1\left(y'' = \sum_2^\infty n(n-1) a_n x^{n-2}\right)$$

$$0 = \sum_0^\infty -a_n x^n + \sum_1^\infty -n a_n x^n + \sum_2^\infty n(n-1) a_n x^{n-2}$$

$$0 = \sum_0^\infty -a_n x^n + \sum_1^\infty -n a_n x^n + \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$$

$$0 = -a_0 + 2a_2 + \sum_1^\infty \left(-a_n - n a_n + (n+2)(n+1) a_{n+2}\right) x^n$$

$$\Rightarrow \quad 0 = -a_0 + 2a_2 \ \& \ 0 = -a_n - n a_n + (n+2)(n+1) a_{n+2} : n \geq 1$$

$$\Rightarrow \quad \boxed{a_2 = \frac{1}{2} a_0 \ \ \& \ \text{for } n \geq 1 : \ a_{n+2} = \frac{a_n}{n+2}}$$

recurrence rel'ns

$$a_3 = \frac{1}{3} a_1 \ ; \ a_4 = \frac{1}{4} a_2 = \frac{1}{4} \cdot \frac{1}{2} a_0 \ ; \ a_5 = \frac{1}{5} a_3 = \frac{1}{5} \cdot \frac{1}{3} a_1$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} a_0 \ ; \ a_7 = \frac{1}{7} a_5 = \frac{1}{7} \cdot \frac{1}{5} \cdot \frac{1}{3} a_1$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \ldots$$

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \frac{1}{48} a_0 x^6 + \frac{1}{105} a_1 x^7 + \ldots$$

$$y = a_0\left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \ldots\right) + a_1\left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \frac{1}{105} x^7 + \ldots\right)$$

The two fcns in parentheses are two linearly indep sol'ns to the orig. prob

15 a

$$2 = y(0) = a_0\left(1 + \tfrac{1}{2}\cdot 0^2 + \tfrac{1}{8}\cdot 0^4 + \cdots\right)$$
$$+ a_1\left(0 + \tfrac{1}{3}\cdot 0^3 + \tfrac{1}{15} 0^5 + \cdots\right) = a_0$$

$$1 = y'(0) = a_0\left(0 + \tfrac{1}{2}\cdot 0^3 + \cdots\right)$$
$$+ a_1\left(1 + 0^2 + \tfrac{1}{3}\cdot 0^4 + \cdots\right) = a_1$$

$$\Rightarrow y = 2\left(1 + \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 + \cdots\right) + 1\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{15}x^5 + \cdots\right)$$

$$y = 2 + x + x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{15}x^5 + \cdots$$

§5.3

11 $y = \sum_0^\infty a_n x^n \Rightarrow y(0) = a_0;\ y'(0) = a_1;\ y''(0) = 2a_2;\ y'''(0) = 6a_3$

$y^{(4)}(0) = 24a_4;\ y^{(5)}(0) = 120a_5;\ y^{(6)}(0) = 720a_6;\ y^{(7)}(0) = 5040a_7$

$$y'' + (\sin x)y = 0 \Rightarrow 2a_2 + 0\cdot a_0 = 0$$

$$y''' + (\cos x)y + (\sin x)y' = 0 \Rightarrow 6a_3 + 1\cdot a_0 + 0\cdot a_1 = 0$$

$$y^{(4)} + (-\sin x)y + 2(\cos x)y' + (\sin x)y'' = 0 \Rightarrow 24a_4 + 0\cdot a_0 + 2\cdot 1\cdot a_1 + 0\cdot 2a_2 = 0$$

next $\Rightarrow 120a_5 + (-1)a_0 + 3\cdot 0\cdot a_1 + 3\cdot 1\cdot 2a_2 + 0\cdot 6a_3 = 0$

next $\Rightarrow 720a_6 + 0\cdot a_0 + 4(-1)a_1 + 6(0)2a_2 + 4(1)6a_3 + 0\cdot 24a_4 = 0$

next $\Rightarrow 5040a_7 + 1\cdot a_0 + 5(0)a_1 + 10(-1)\cdot 2a_2 + 10\cdot 0\cdot 6a_3 + 5\cdot 1\cdot 24a_4 + 0\cdot 120a_5 = 0$

$a_2 = 0;\ a_3 = -\frac{1}{6}a_0;\ a_4 = -\frac{1}{12}a_1;\ a_5 = a_0\cdot\frac{1}{120} - \frac{1}{20}a_2 = \frac{1}{120}a_0$

$a_6 = \frac{1}{180}a_1 - \frac{1}{30}a_3 = \frac{1}{180}a_0 + \frac{1}{180}a_1;\ a_7 = -\frac{1}{5040}a_0 + \frac{1}{252}a_2 - \frac{1}{42}a_4 = -\frac{1}{5040}a_0 + \frac{1}{504}a_1$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$$

$$y = a_0 + a_1 x + 0x^2 - \frac{1}{6}a_0 x^3 - \frac{1}{12}a_1 x^4 + \frac{1}{120}a_0 x^5 + \left(\frac{1}{180}a_0 + \frac{1}{180}a_1\right)x^6 + \left(-\frac{1}{5040}a_0 + \frac{1}{504}a_1\right)x^7 + \cdots$$

$$y = a_0\left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7 + \cdots\right)$$
$$+ a_1\left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \cdots\right)$$

The coeff of y'', 1, is never 0.
The coeffs of y' & y, 0 & $\sin x$, are always well-behaved. Therefore $y'' + (\sin x)y = 0$ has no singular points, and ROC $= +\infty$.