ERIC WEISSTEIN'S world of MATHEMATICS

SEARCH
go

INDEX BY SUBJECT
- Algebra
- Applied Mathematics
- Calculus and Analysis
- Discrete Mathematics
- Foundations of Mathematics
- Geometry
- History and Terminology
- Number Theory
- Probability and Statistics
- Recreational Mathematics
- Topology

ALPHABETICAL INDEX
ABOUT THIS SITE
FAQs
WHAT'S NEW
RANDOM ENTRY
BE A CONTRIBUTOR
SIGN THE GUESTBOOK
EMAIL COMMENTS

Stirling's Approximation

Stirling's approximation gives an approximate value for the factorial function $n!$ or the gamma function $\Gamma(n)$ for $n \gg 1$. The approximation can most simply be derived for n an integer by approximating the sum over the terms of the factorial with an integral, so that

$$\begin{aligned} \ln n! &= \ln 1 + \ln 2 + \ldots + \ln n = \sum_{k=1}^{n} \ln k \approx \int_1^n \ln x \, dx \\ &= [x \ln x - x]_1^n = n \ln n - n + 1 \approx n \ln n - n. \end{aligned} \quad (1)$$

The equation can also be derived using the integral definition of the factorial,

$$n! = \int_0^\infty e^{-x} x^n \, dx. \quad (2)$$

Note that the derivative of the logarithm of the integrand can be written

$$\frac{d}{dx} \ln(e^{-x} x^n) = \frac{d}{dx}(n \ln x - x) = \frac{n}{x} - 1. \quad (3)$$

The integrand is sharply peaked with the contribution important only near $x = n$. Therefore, let $x \equiv n + \xi$ where $\xi \ll n$, and write

$$\ln(x^n e^{-x}) = n \ln x - x = n \ln(n + \xi) - (n + \xi). \quad (4)$$

Now,

$$\begin{aligned} \ln(n + \xi) &= \ln\left[n\left(1 + \frac{\xi}{n}\right)\right] = \ln n + \ln\left(1 + \frac{\xi}{n}\right) \\ &= \ln n + \frac{\xi}{n} - \frac{1}{2}\frac{\xi^2}{n^2} + \ldots, \end{aligned} \quad (5)$$

so

$$\ln(x^n e^{-x}) = n \ln(n + \xi) - (n + \xi)$$

$$\begin{aligned} &= n\ln n + \xi - \frac{1}{2}\frac{\xi^2}{n} - n - \xi + \ldots \\ &= n\ln n - n - \frac{\xi^2}{2n} + \ldots. \end{aligned} \tag{6}$$

Taking the exponential of each side then gives

$$x^n e^{-x} \approx e^{n\ln n} e^{-n} e^{-\xi^2/2n} = n^n e^{-n} e^{-\xi^2/2n}. \tag{7}$$

Plugging into the integral expression for $n!$ then gives

$$n! \approx \int_{-n}^{\infty} n^n e^{-n} e^{-\xi^2/2n}\, d\xi \approx n^n e^{-n} \int_{-\infty}^{\infty} e^{-\xi^2/2n}\, d\xi. \tag{8}$$

Evaluating the integral gives

$$n! \approx n^n e^{-n}\sqrt{2\pi n}, \tag{9}$$

$$= \sqrt{2\pi}\, n^{n+1/2} e^{-n} \tag{10}$$

(Wells 1986, p. 45). Taking the logarithm of both sides then gives

$$\ln n! \approx n\ln n - n + \tfrac{1}{2}\ln(2\pi n) = (n + \tfrac{1}{2})\ln n - n + \tfrac{1}{2}\ln(2\pi). \tag{11}$$

This is Stirling's series with only the first term retained and, for large *n*, it reduces to Stirling's approximation

$$\ln n! \approx n\ln n - n. \tag{12}$$

Taking successive terms of $\lfloor n^n/n! \rfloor$, where $\lfloor x \rfloor$ is the floor function, gives the sequence 1, 2, 4, 10, 26, 64, 163, 416, 1067, 2755, ... (Sloane's A055775).

Stirling's approximation can be extended to the double inequality

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n+1/(12n+1)} < n! < \sqrt{2\pi}\, n^{n+1/2} e^{-n+1/(12n)} \tag{13}$$

(Robbins 1955, Feller 1968).

Gosper has noted that a better approximation to $n!$ (i.e., one which approximates the terms in Stirling's series instead of truncating them) is given by

$$n! \approx \sqrt{(2n + \tfrac{1}{3})\pi}\, n^n e^{-n}. \tag{14}$$

This also gives a much closer approximation to the factorial of 0, $0! = 1$, yielding $\sqrt{\pi/3} \approx 1.02333$ instead of 0 obtained with the conventional Stirling approximation.

SEE ALSO: Stirling's Series

References

Feller, W. "Stirling's Formula." §2.9 in *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd ed.* New York: Wiley, pp. 50-53, 1968.

Robbins, H. "A Remark of Stirling's Formula." *Amer. Math. Monthly* **62**, 26-29, 1955.

Sloane, N. J. A. Sequences A055775 in "An On-Line Version of the Encyclopedia of Integer Sequences." http://www.research.att.com/~njas/sequences/eisonline.html

Stirling, J. *Methodus differentialis.* 1730.

Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers.* Middlesex, England: Penguin Books, p. 45, 1986.

Whittaker, E. T. and Robinson, G. "Stirling's Approximation to the Factorial." §70 in *The Calculus of Observations: A Treatise on Numerical Mathematics, 4th ed.* New York: Dover, pp. 138-140, 1967.