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SEARCH
go
INDEX BY SUBJECT
- Algebra
- Applied Mathematics
- Calculus and Analysis
- Discrete Mathematics
- Foundations of Mathematics
- Geometry
- History and Terminology
- Number Theory
- Probability and Statistics
- Recreational Mathematics
- Topology

ALPHABETICAL INDEX
ABOUT THIS SITE
FAQs
WHAT'S NEW
RANDOM ENTRY
BE A CONTRIBUTOR
SIGN THE GUESTBOOK
EMAIL COMMENTS

Stirling's Series

The asymptotic series for the gamma function is given by

$$\Gamma(z) \sim e^{-z} z^{z-1/2} \sqrt{2\pi} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \ldots\right) \quad (1)$$

(Sloane's A001163 and A001164).

The coefficient a_n of z^{-n} can given explicitly by

$$a_n = \sum_{k=1}^{2n} (-1)^k \frac{d_3(2n+2k, k)}{2^{n+k}(n+k)!}, \quad (2)$$

where $d_3(n, k)$ is the number of permutations of *n* with *k* cycles all of which are ≥ 3 (Comtet 1974, p. 267). Another formula for the a_n s is given by the recurrence relation

$$b_n = \frac{1}{n+1}\left(b_{n-1} - \sum_{k=2}^{n-1} k a_k a_{n+1-k}\right), \quad (3)$$

with $b_0 = b_1 = 1$, then

$$a_n = (2n+1)!! b_{2n+1}, \quad (4)$$

where $x!!$ is the double factorial (Borwein and Corless 1999).

The series for $z!$ is obtained by adding an additional factor of *z*,

$$\begin{aligned} z! &= \Gamma(z+1) \\ &= e^{-z} z^{z+1/2} \sqrt{2\pi} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \ldots\right). \end{aligned} \quad (5)$$

The expansion of $\ln \Gamma(z)$ is what is usually called Stirling's series. It is given by the simple analytic expression

$$\ln \Gamma(z) = \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \quad (6$$

$$= \tfrac{1}{2}\ln(2\pi) + (z - \tfrac{1}{2})\ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots, \quad (7$$

where B_n is a Bernoulli number.

SEE ALSO: Bernoulli Number, Cycle (Permutation), *K*-Function, Stirling's Approximation

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