Subgrid Upscaling Preconditioners for Darcy Flow

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Problem:

$$\nabla \cdot \mathbf{u} = q$$

$$\mathbf{u} = -\frac{\kappa}{\mu}(\nabla p - \rho \mathbf{g})$$

Approximate flow equations using a cell-centered finite difference method (CCFD) for the pressures.

Difficulties:

Heterogeneity in the permeability and its fine-scale resolution makes the problem ill-conditioned and computationally expensive to solve.

The Problem in Mixed Variational Form:

Let

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} \in H(\mathrm{div}, \Omega) \mid \mathbf{v} \cdot \nu = 0 \text{ on } \partial\Omega\}, \\ W &= L^2(\Omega). \end{aligned}$$

Find $\mathbf{u} \in \mathbf{V}$ and $p \in W$ such that

$$\begin{aligned} &(\nabla \cdot \mathbf{u}, w) = (q, w) && \forall w \in W, \\ &(\mu\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\rho\mathbf{g}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}. \end{aligned}$$

The Finite Element Approximation:

For a fine-scale problem, assume κ is piecewise constant on a grid with spacing h.

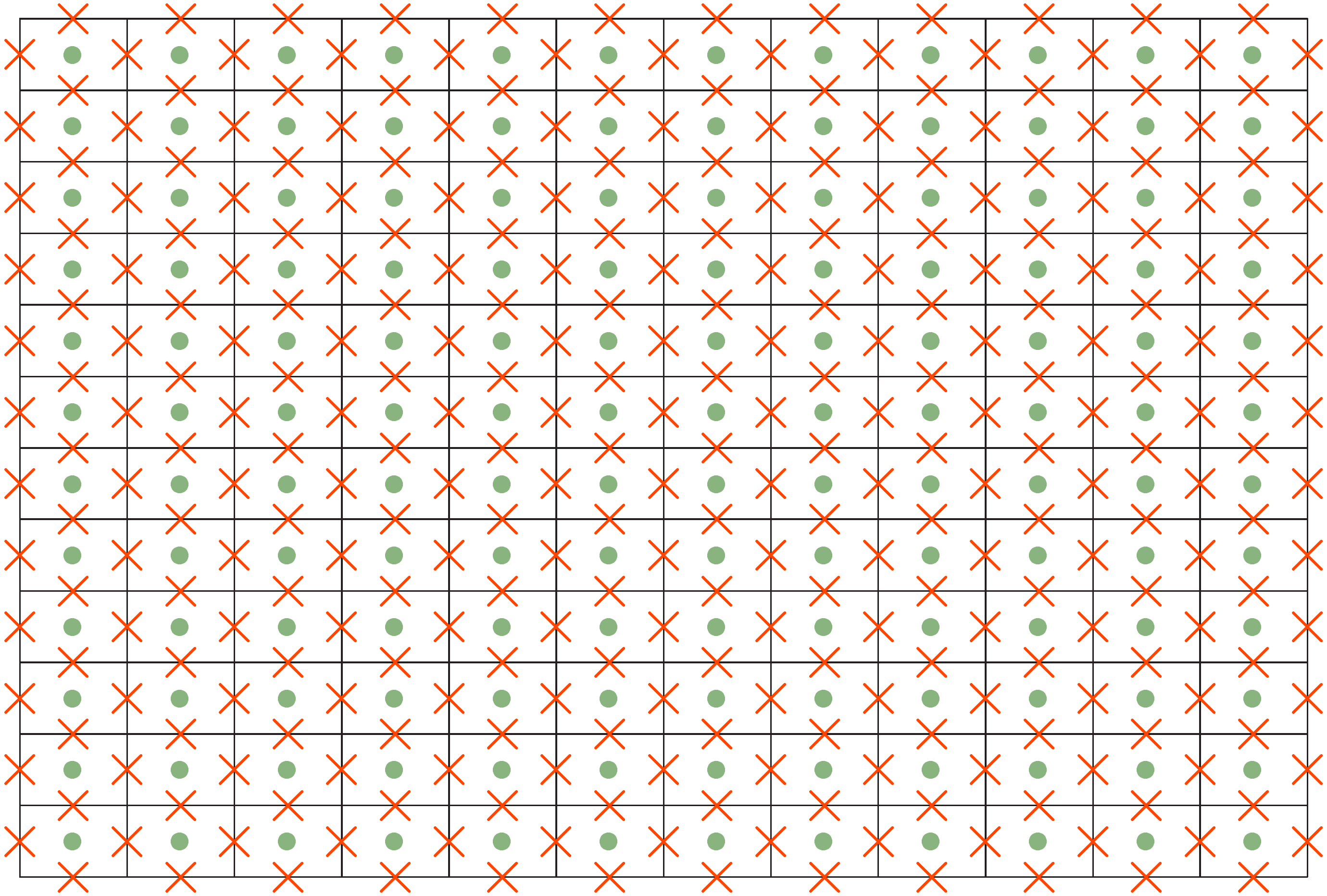
Find $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{V}$ and $p_h \in W_h \subset W$ such that

$$\begin{aligned}
&(\nabla \cdot \mathbf{u}_h, w_h) = (q, w_h) && \forall w_h \in W_h, \\
&(\mu\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = (\rho\mathbf{g}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned}$$

where $W_h \times \mathbf{V}_h$ are Raviart-Thomas-Nedelec elements of order zero (RT0) on the given grid.

Using quadrature — the trapezoidal rule — to compute the $(\mu\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h)$ term is equivalent to solving the original problem using CCFD.

Degrees of freedom for 2-D RT0 elements on a fine rectangular grid:



× **Velocity**
• **Pressure**

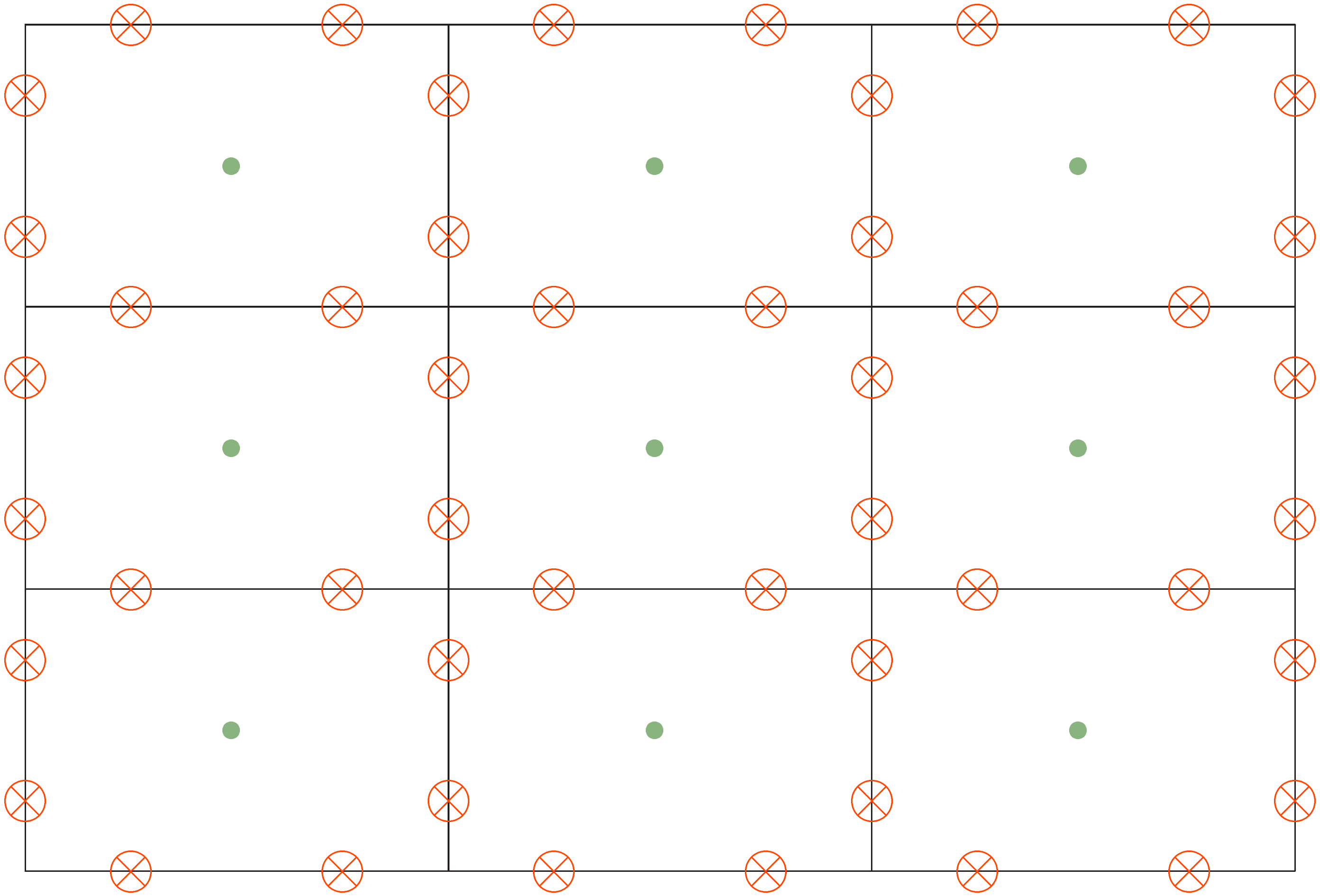
If instead we desire a coarse-scale approximation, we can find $\mathbf{u}_H \in \mathbf{V}_H \subset \mathbf{V}$ and $p_H \in W_H \subset W$ such that

$$\begin{aligned} &(\nabla \cdot \mathbf{u}_H, w_H) = (q, w_H) && \forall w_H \in W_H, \\ &(\mu\kappa^{-1}\mathbf{u}_H, \mathbf{v}_H) - (p_H, \nabla \cdot \mathbf{v}_H) = (\rho\mathbf{g}, \mathbf{v}_H) && \forall \mathbf{v}_H \in \mathbf{V}_H. \end{aligned}$$

where $W_H \times \mathbf{V}_H$ are RT0 elements (or perhaps Brezzi-Douglas-Duràn-Fortin order one (BDDF1) elements) on a grid with spacing $H = Ch$.

With the RT0 elements, this is equivalent to using a κ which is piecewise constant on the H grid, and is the harmonic average of the fine-scale κ.

Degrees of freedom for 2-D BDDF1 elements on a coarse rectangular grid:



⊗ **Velocity (linear)**

• **Pressure**

Idea:

Use a multiscale approximation as the "coarse" level in a two-level iterative scheme to precondition a fine-scale CCFD approximation. By upscaling the original problem (rather than simply coarsening it as in multigrid (MG)), we retain more of the fine scale information in the computation thereby (hopefully) obtaining a better approximation and a faster scheme.

Our subgrid upscaling technique uses the mixed finite element method (MFEM) with a variational multiscale technique.

The Upscaling Technique:

Decompose $\mathbf{V}$ and W (using a chosen coarse grid) so that:

(1) we do not lose any information, $\mathbf{V} = \mathbf{V}_c \oplus \delta\mathbf{V}$, $W = W_c \oplus \delta W$;

(2) mass is conserved, $\nabla \cdot \mathbf{V}_c = W_c$, $\nabla \cdot \delta\mathbf{V} = \delta W$;

(3) the fine-scale velocities are locally supported over the coarse grid, $\delta\mathbf{V} \cdot \nu = 0$ on the boundary of each coarse cell.

Condition (3) allows us to disconnect the subgrid problems from one another — but not from the coarse grid problem. We employ a Green's or influence function approach to effect a forward elimination of the subgrid problems into the coarse problem, and — once the coarse problem has been solved — a back substitution to recover the subgrid solutions. The two can then be recombined to obtain a fine scale solution.

Summary of Advantages:

- Each subgrid problem is independent from the others; they can be solved in parallel with no communication needed between coarse cells.
- Each subgrid problem also has many fewer degrees of freedom than the whole problem (by about a factor of the number of coarse grid cells).
- Each subgrid problem has the same LHS and many different RHS creating an opportunity for efficiency.
- The coarse problem has many fewer degrees of freedom than the whole problem (by about a factor of the number of fine cells chosen per coarse cell).
- If the coarse grid spacing is a small multiple of the fine grid spacing, the work in solving the upscaled problem is nearly all done in solving the coarse problem.
- There is only one coarse grid (not many as in MG).

Approximation of the scale-separated problem:

We use Raviart-Thomas-Nedelec order zero elements to approximate δW and $\delta \mathbf{V}$, denoted δW_h and $\delta \mathbf{V}_h$, and BDDF order one elements for W_c and $\mathbf{V}_c$, denoted W_H and $\mathbf{V}_H$.

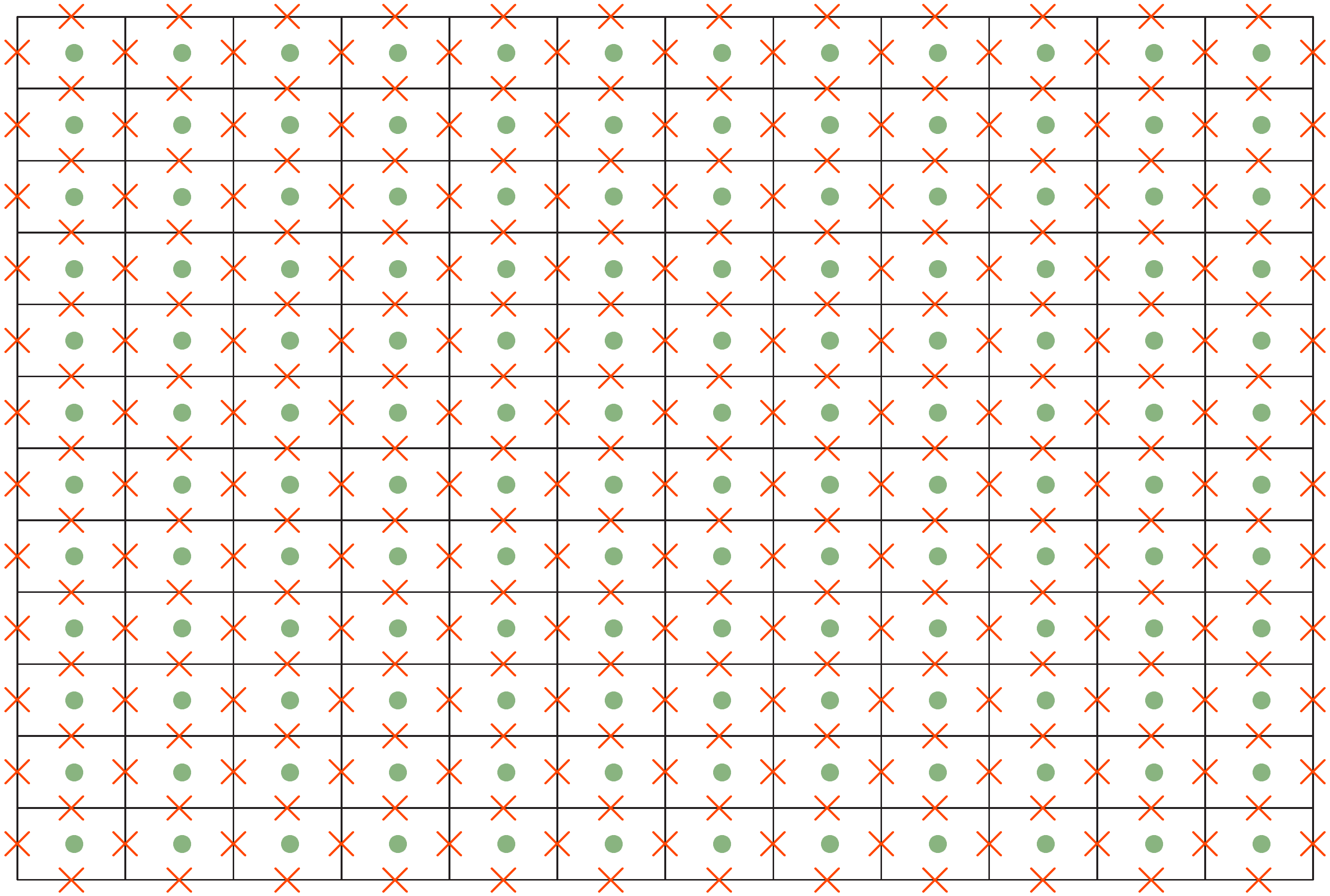
W_H: **functions which are piecewise constant on coarse cells**

δW_h: **functions which are piecewise constant on fine cells, but must have a zero average over each coarse cell**

$\mathbf{V}_H$: **vector-valued functions where the components normal to coarse cell faces are continuous across those faces, and vary linearly along those faces**

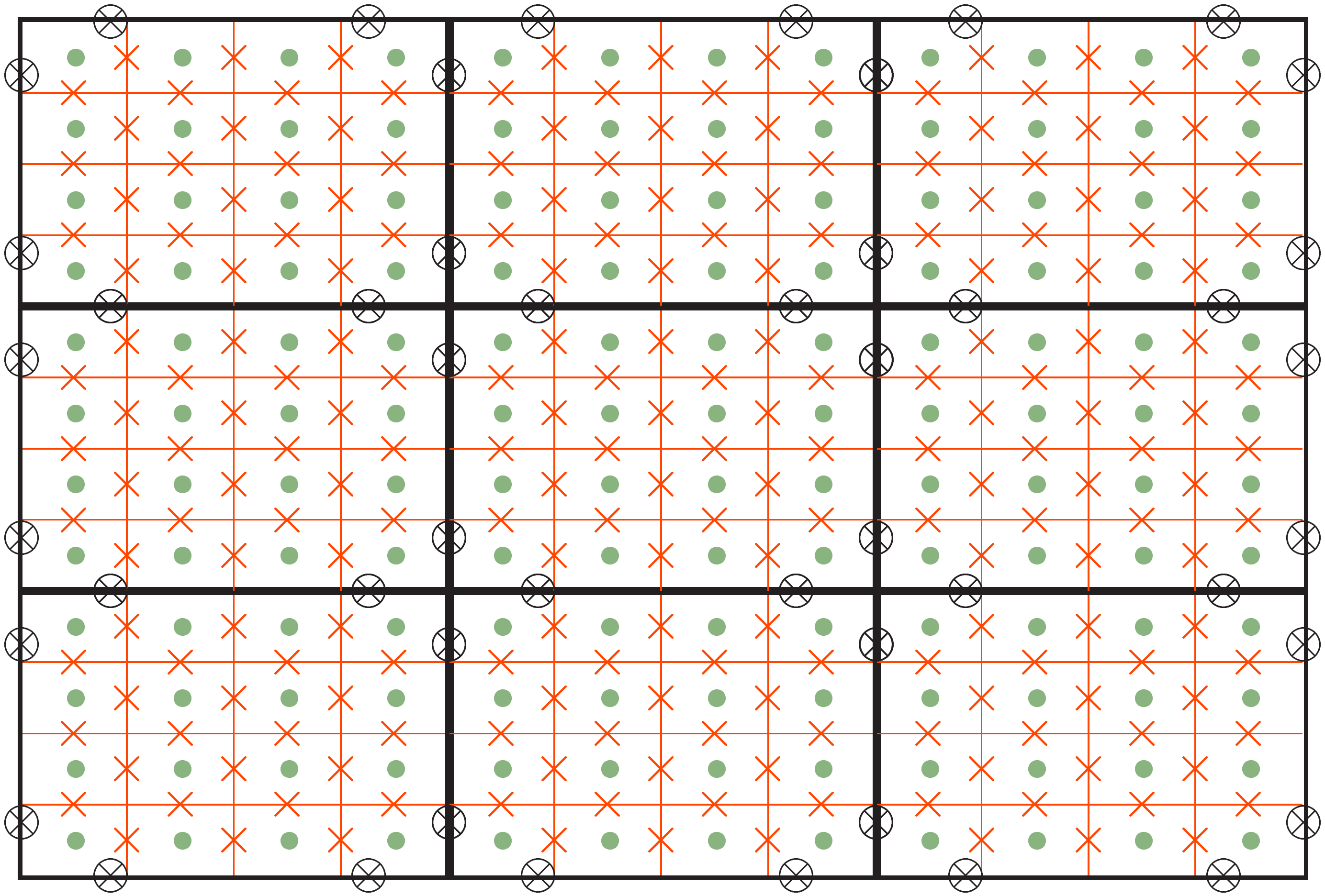
$\delta \mathbf{V}_h$: **vector-valued functions where the components normal to fine cell faces are continuous across those faces, and are constant along those faces; the normal components across coarse cell faces are zero**

Degrees of freedom for 2-D RT0 elements on a fine rectangular grid:



× Velocity
● Pressure

Degrees of freedom for 2-D RT0/BDDF1 elements on an "upscaled" rectangular grid:



$\otimes$ Coarse velocity (linear) $\times$ Subgrid velocity

$\bullet$ Pressure

Error Estimates for the Approximation:

No closure assumption is made regarding the permeability.

Accuracy is simply a question of approximation theory — how well does $\mathbf{V}_{H,h}$ approximate V? — and not how much the physics being changed.

Our closure assumption can be said to be that all flux across a coarse element face is due to the coarse scale functions. To compensate for the coarseness of this restriction, higher order accurate basis functions have been employed.

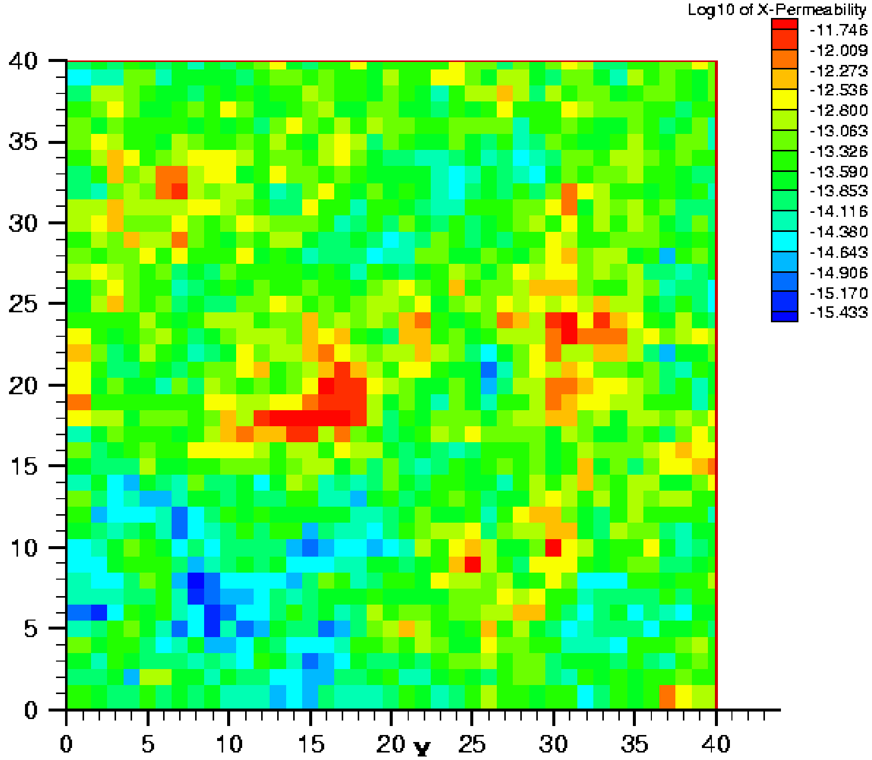
For the BDDF1/RT0 elements:

$$\|\mathbf{u} - \mathbf{u}_{H,h}\|_0 \leq C\|\mathbf{u}\|_2 H^2 = O(H^2)$$
$$\|p - p_{H,h}\|_0 \leq C\{[\|p\|_1 + \|\nabla \cdot \mathbf{u}\|_1 h]h + \|\mathbf{u}\|_2 H^3\} = O(h + H^3)$$

Some Practical Results:

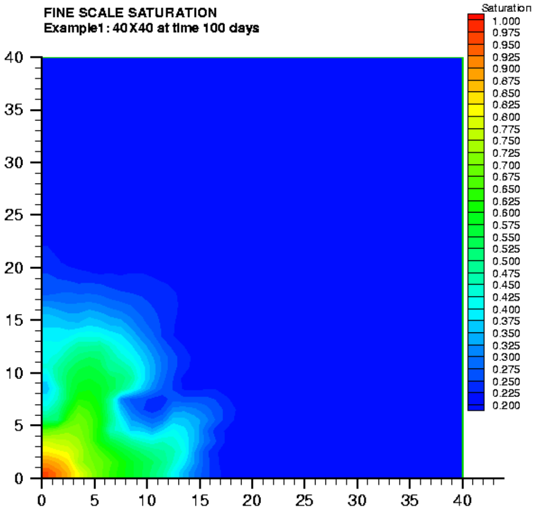
Quarter Five-spot Oil Reservoir Water Flood

Logarithm of the Permeability Field

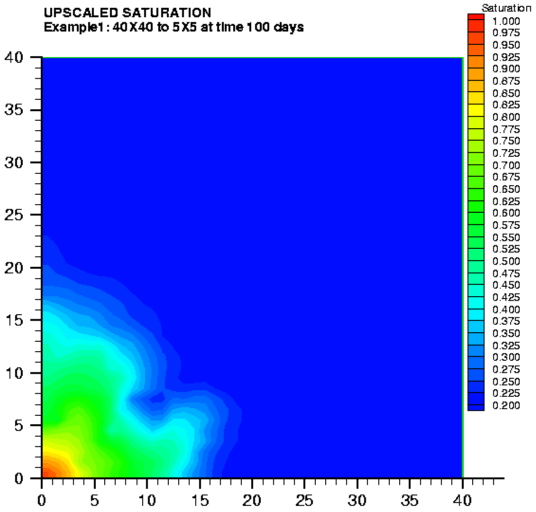


Fine 40×40

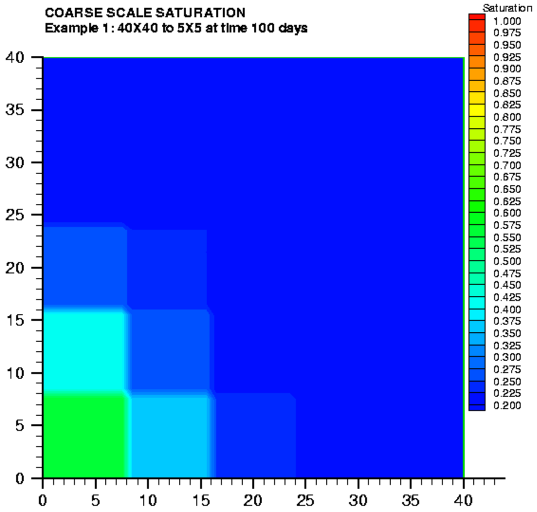
Water saturation contours at 100 days



Fine 40×40

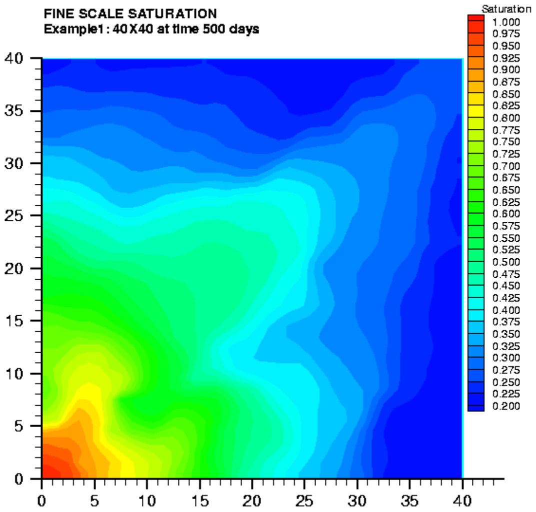


Fully upscaled to 5×5

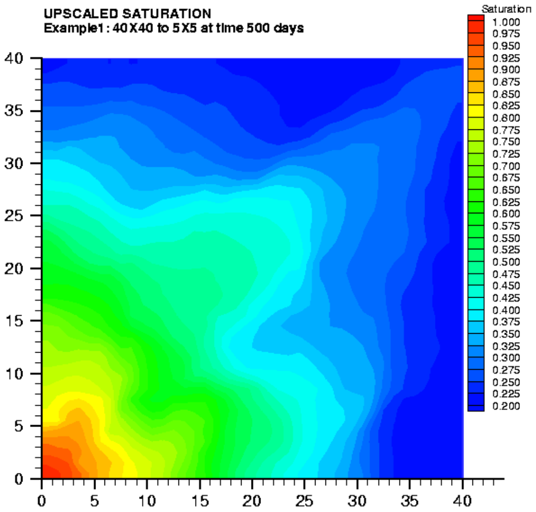


Coarse 5×5

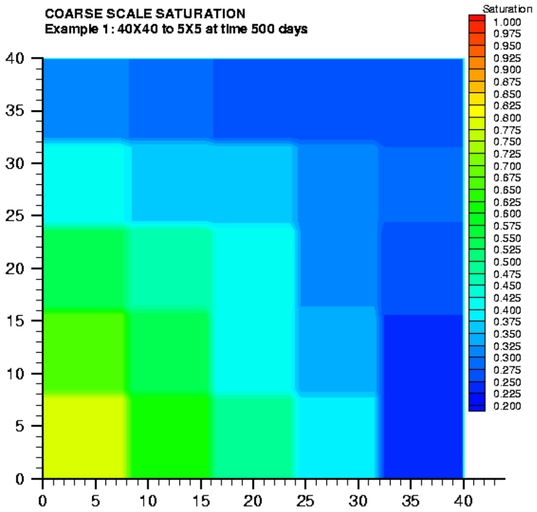
Water saturation contours at 500 days



Fine 40×40



Fully upscaled to 5×5



Coarse 5×5

Adaptation to a Two-Level Scheme:

A: matrix which describes the interactions among cells

$\mathbf{x}$: pressure unknowns

$\mathbf{b}$: information from sources, boundary terms, and gravity

Then the problem is to solve the linear system $A\mathbf{x} = \mathbf{b}$.

$\mathbf{x}_i$: ith guess at pressure unknowns

$\mathbf{r}_i$: ith-stage residual $\mathbf{b} - A\mathbf{x}_i$

$\mathbf{c}_i$: ith-stage smoothing corrector

B_{J}: diagonal elements of A used in Jacobi smoothing step

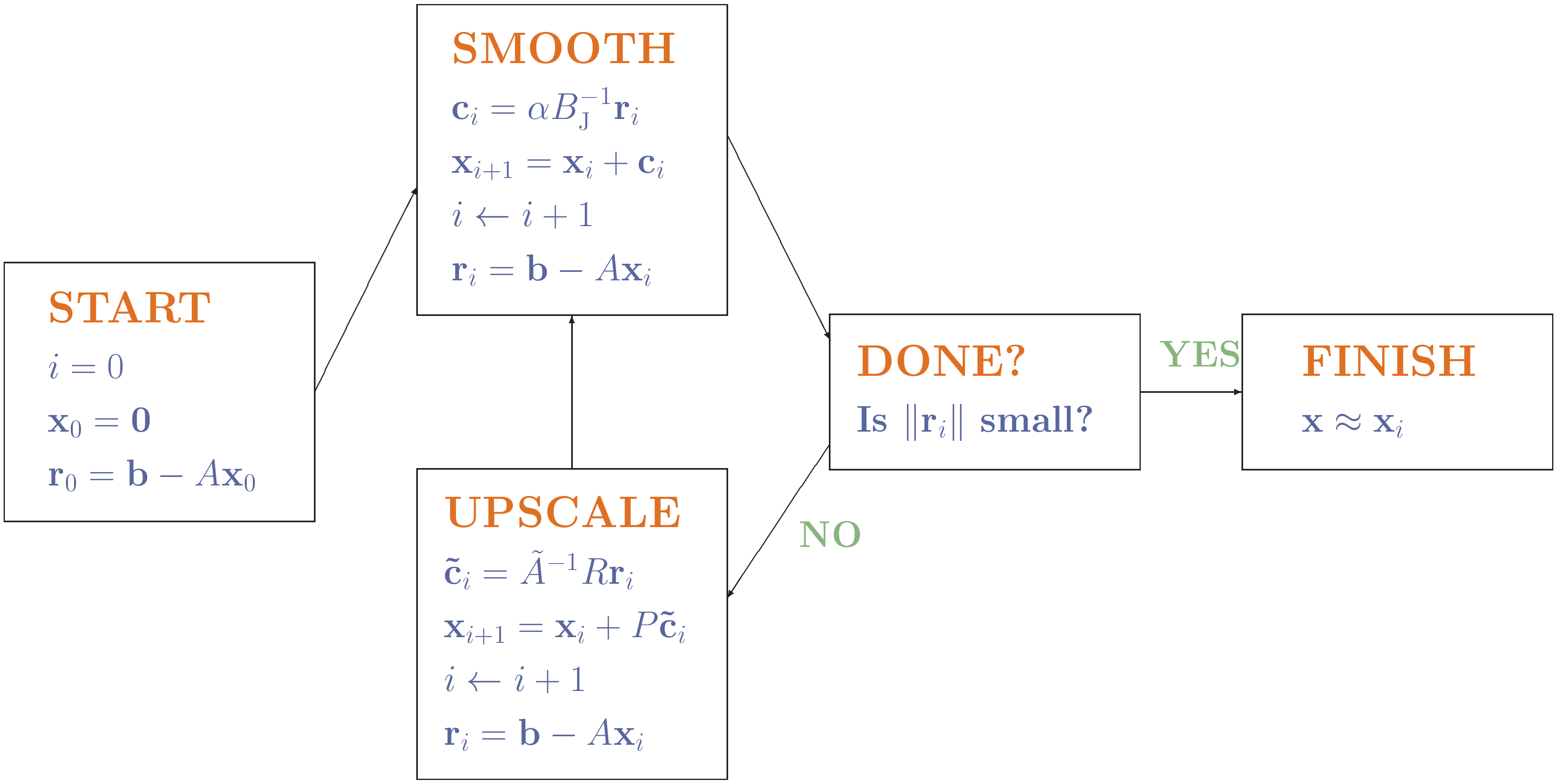
$\tilde{A}$: matrix for the corresponding upscaling MFEM problem

$\tilde{\mathbf{c}}_i$: ith-stage upscaling corrector (pressure and velocity unknowns)

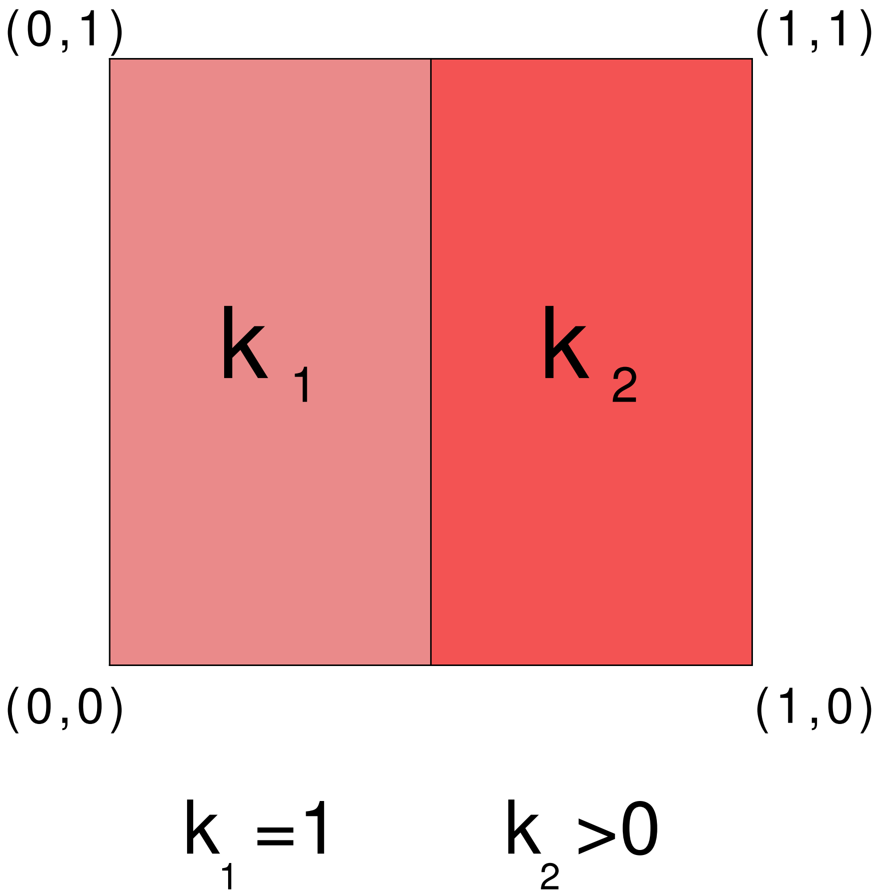
R: restriction operator $R : W_h \to W_{H,h} \times \mathbf{0}$

P: prolongation operator $P : W_{H,h} \times \mathbf{V}_{H,h} \to W_h$

The two-level scheme can be described schematically:



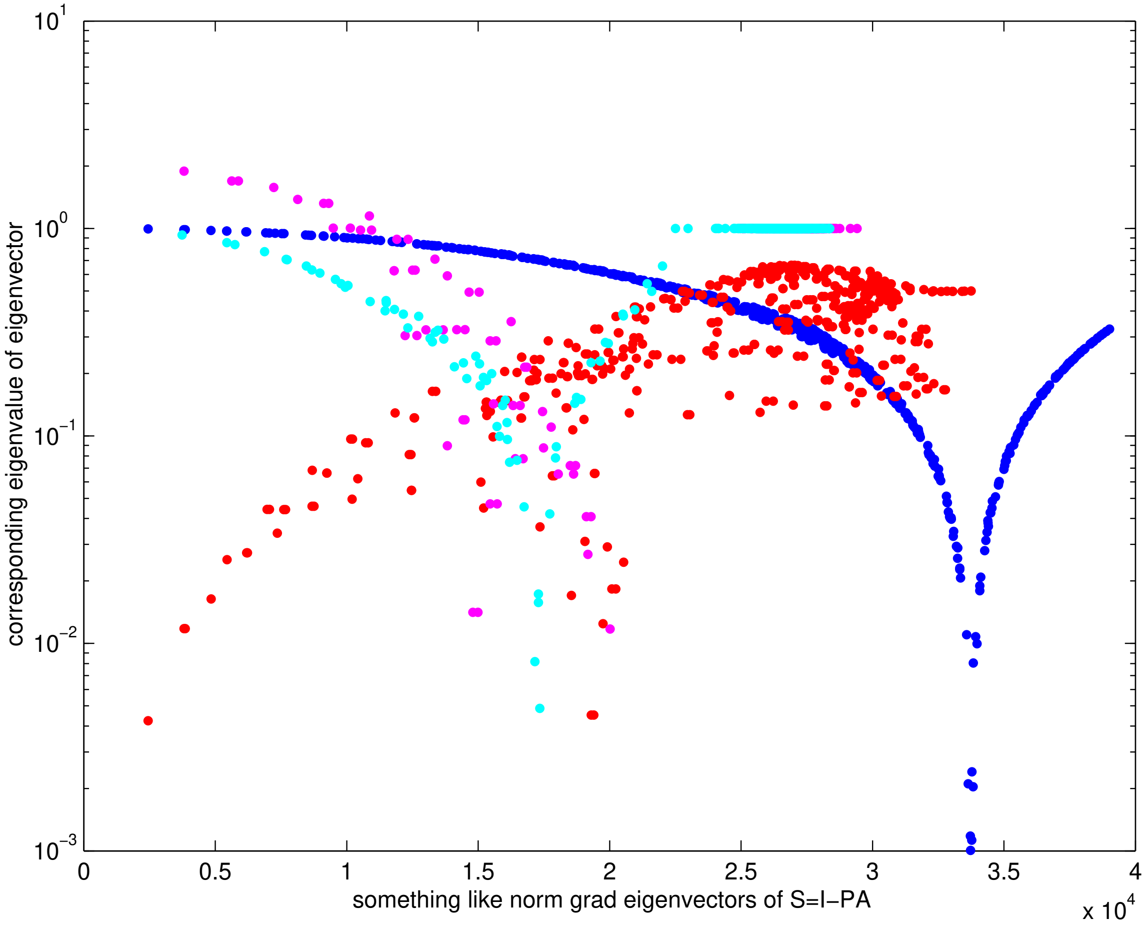
A Simple Test Case:



A Permeability Field

A “Typical” Result:

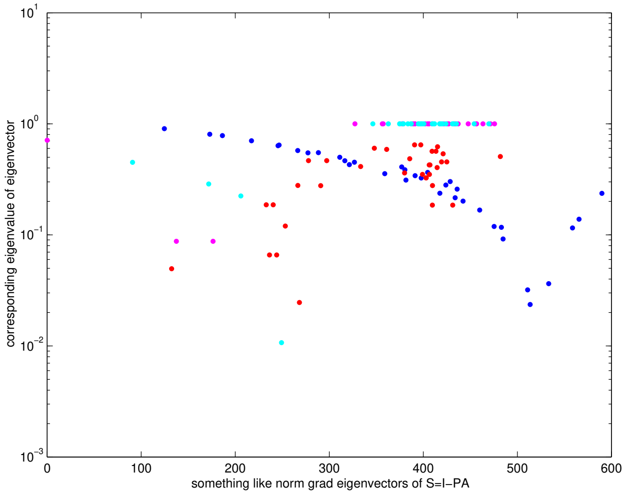
Eigenstructure of Smoother and Upscaler



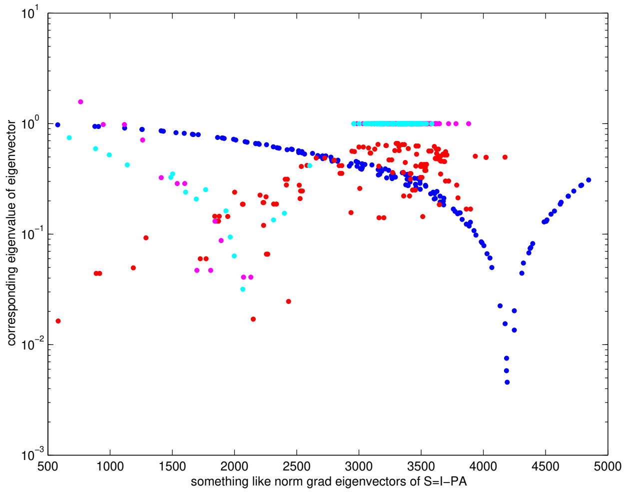
• Coarse Smoothing Preconditioner • Coarse Exact Preconditioner

• Fine Smoothing Preconditioner • Subgrid Upscaling Preconditioner

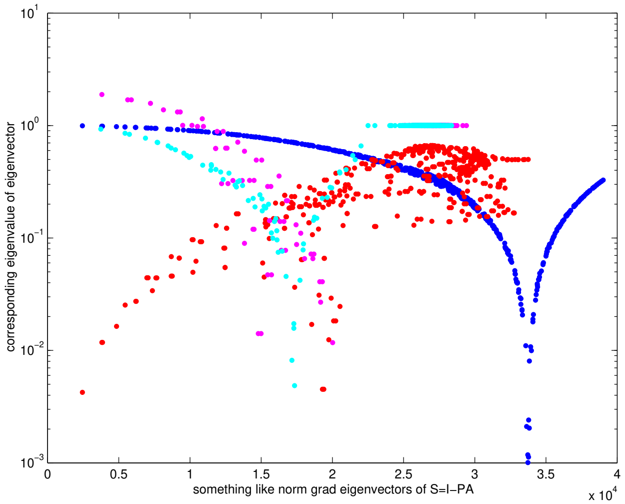
Vary h ; Fix $H/h = 3$; Fix $k_2 = 0.1$



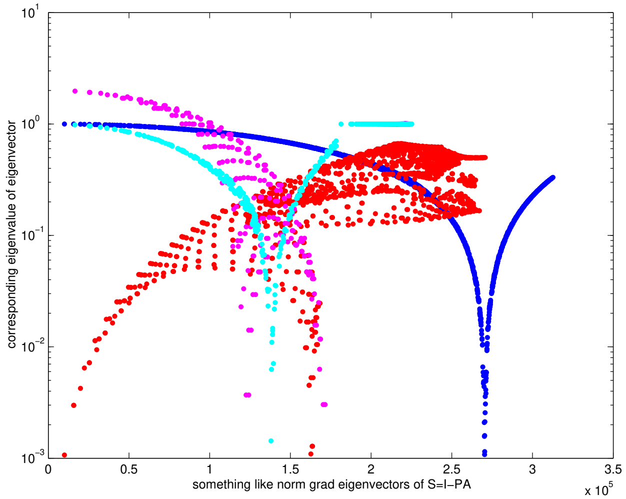
6×6



12×12

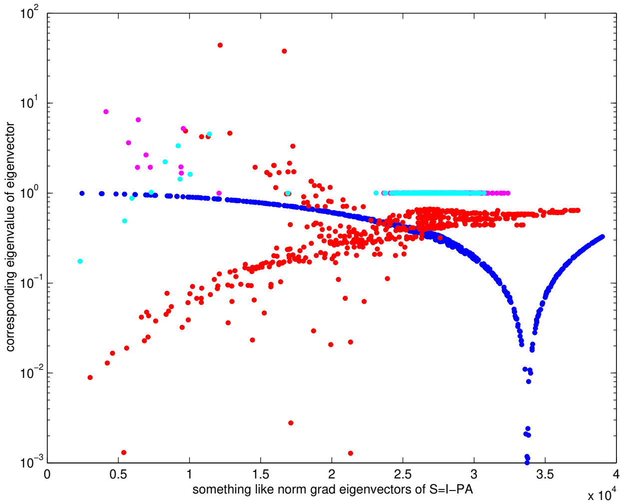


24×24

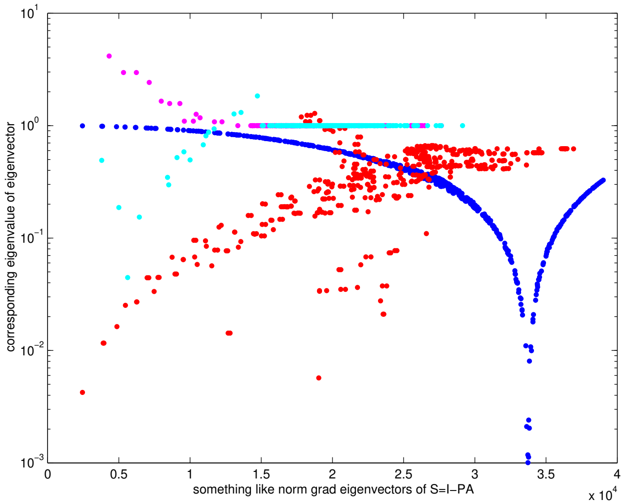


48×48

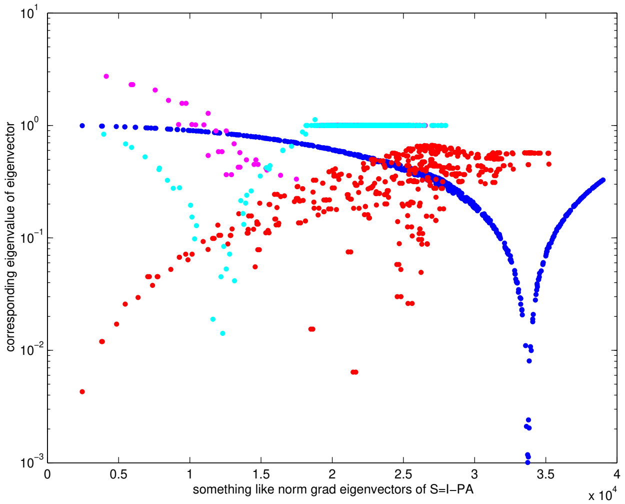
Fix $h = 1/24$; **Vary** H/h ; **Fix** $k_2 = 0.1$



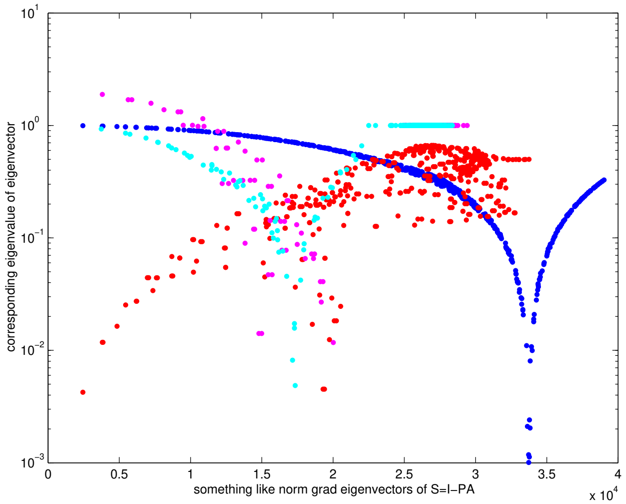
$H/h = 8$



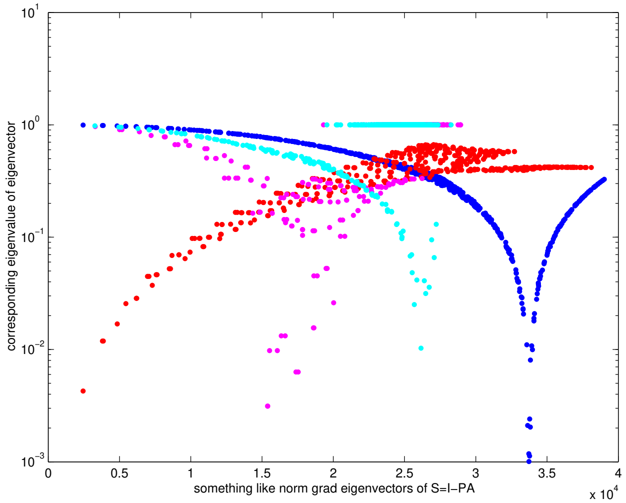
$H/h = 6$



$H/h = 4$

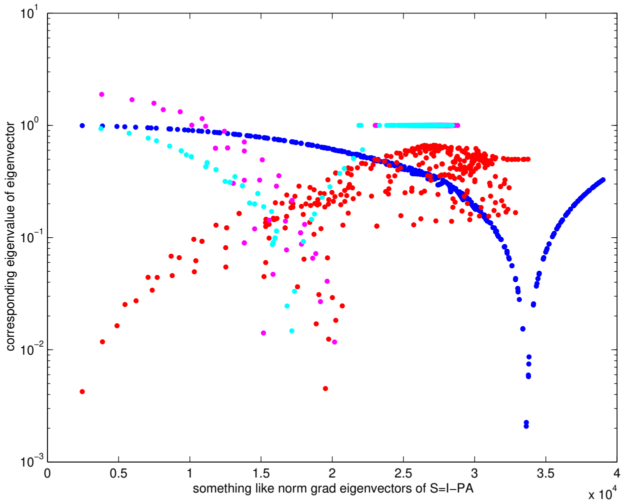


$H/h = 3$

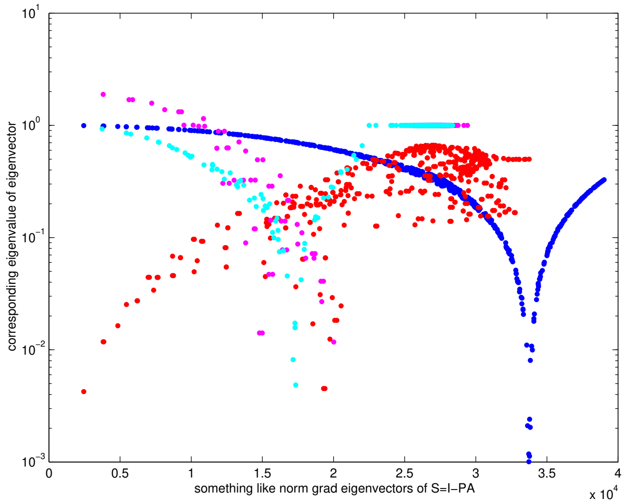


$H/h = 2$

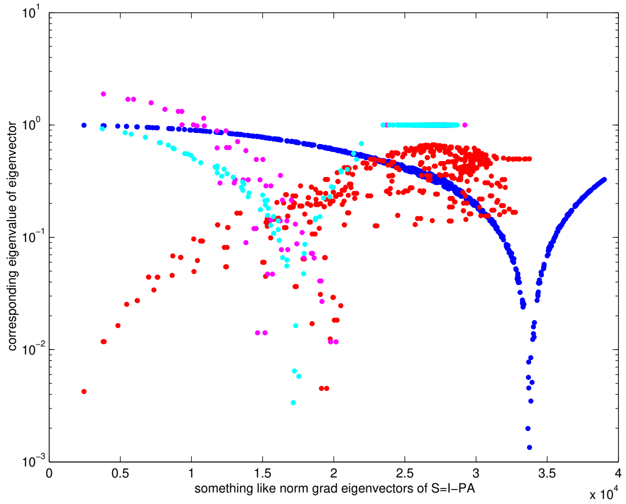
Fix $h = 1/24$; **Fix** $H/h = 3$; **Vary** k_2



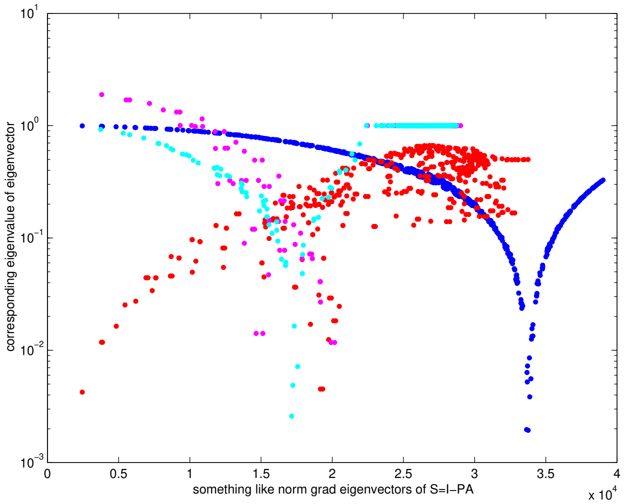
$k_2 = 1e - 0$



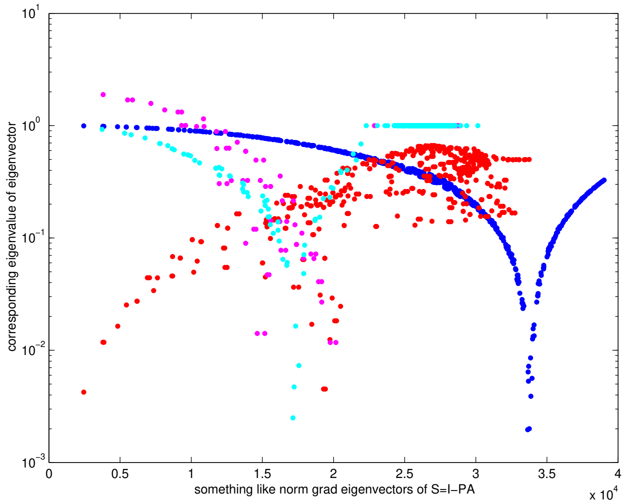
$k_2 = 1e - 1$



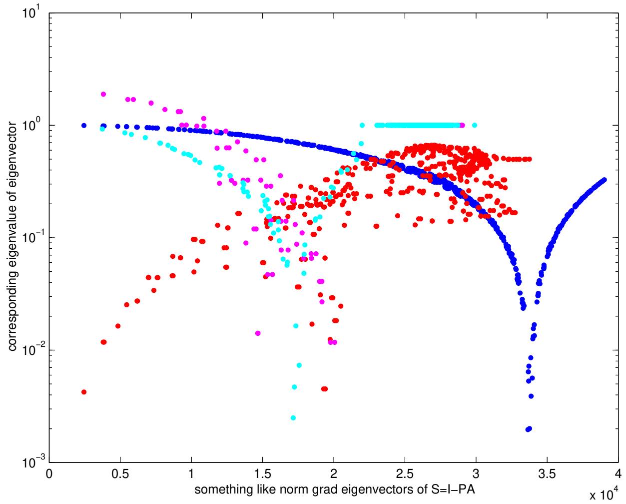
$k_2 = 1e - 2$



$k_2 = 1e - 3$



$k_2 = 1e - 4$



$k_2 = 1e - 5$

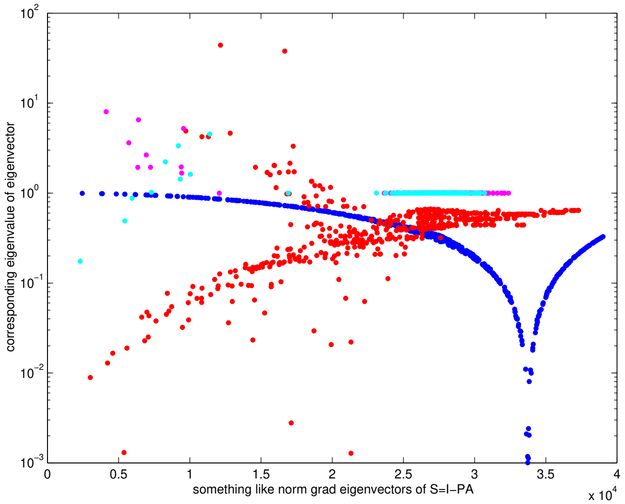
Some Conclusions:

- This scheme will have excellent performance on smooth problems. This observation has been borne out in several other tests with more complicated permeability fields.
- This scheme will perform just like the simple test case with more complicated heterogeneous permeability fields. This observation, too, has been borne out in practice.
- Although we cannot prove it yet, this simple test case suggests:
  - performance of the scheme depends only on H/h and geometric factors,
  - performance is independent of h and k,
  - and we can adjust the number of smoothing steps to make the two-level scheme a contraction for any H/h

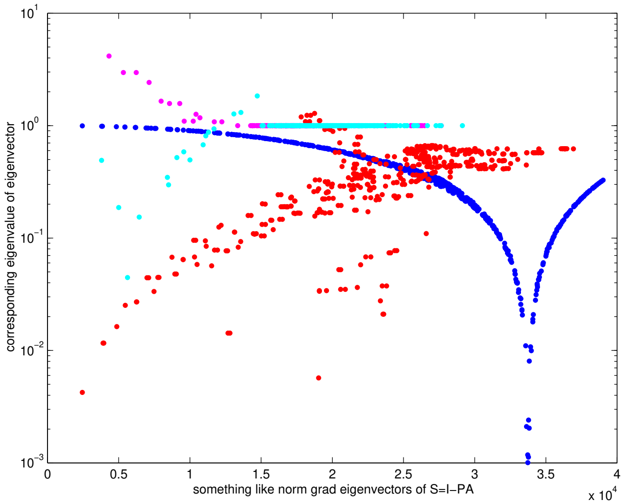
Preliminary Results:

- **convergence factor for one V-cycle is independent of the permeability and dependent only on the relative sizes of the coarse and fine grids (like multigrid)**
- $\mathrm{O}(n)$ **operations for both multigrid and this scheme (versus** $\mathrm{O}(n \log n)$ **for Jacobi preconditioned conjugate gradients)**
- **a bit easier to apply to general meshes than multigrid (only one coarse mesh, not several)**
- **wall-clock time speed-up of a factor of three or more over using Jacobi preconditioned CG to solve a hybrid MFEM on the fine scale for a few test cases (both smooth and heterogeneous permeabilities)**
- **by replacing** κ **with an averaged value in the coarse, upscaled equations, more accurate coarse flow information is sometimes obtained**

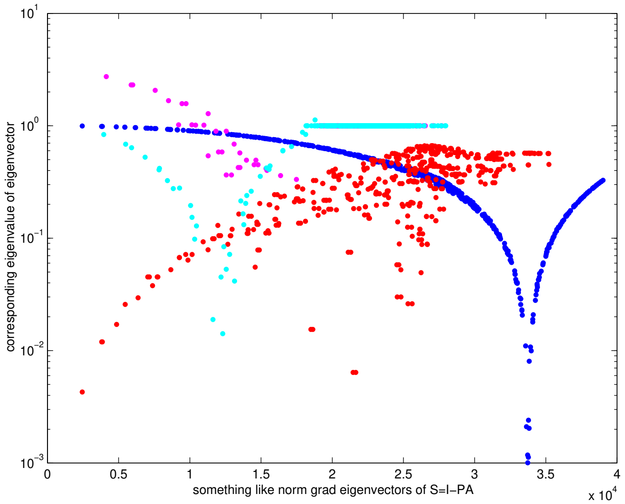
Fix $h = 1/24$; **Vary** H/h ; **Fix** $k_2 = 0.1$



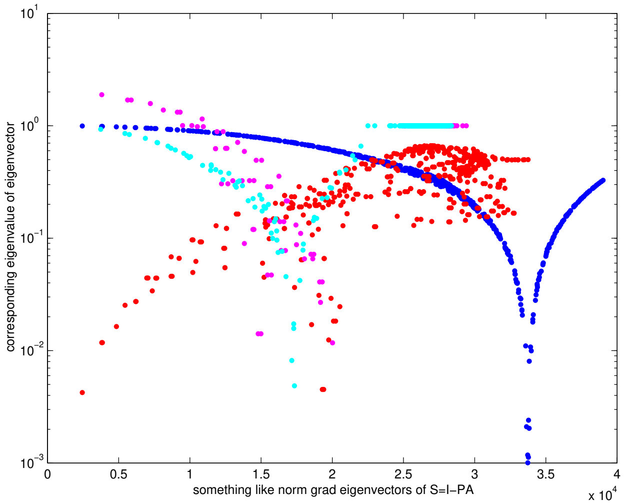
$H/h = 8$



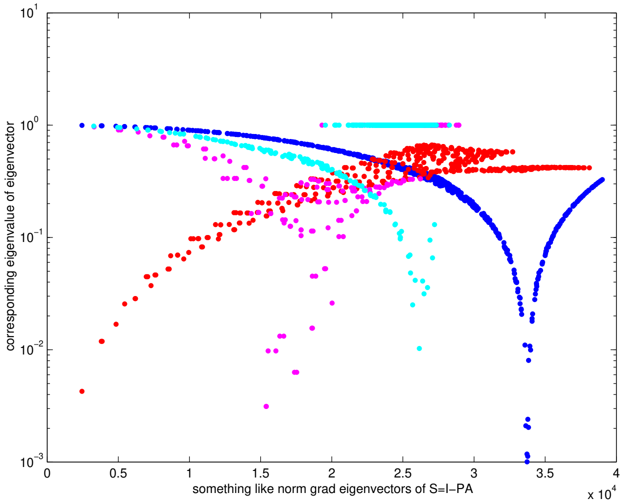
$H/h = 6$



$H/h = 4$



$H/h = 3$



$H/h = 2$