

7.7

19, 20, 25: Differentiate the given expression:

①⑨ $y = \tan^{-1}(x+1)$

$$y' = \frac{1}{\underbrace{(x+1)^2 + 1}_{\frac{d}{du}(\tan^{-1}u)}} \cdot \underbrace{(1+0)}_{\frac{d}{dx}(x+1)} = \frac{1}{x^2 + 2x + 2}$$

②⑩ $y = \tan^{-1}\sqrt{x}$

$$y' = \frac{1}{(\sqrt{x})^2 + 1} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}(x+1)}$$

②⑤ $y = (\sin^{-1}x)^2$

$$y' = 2(\sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}}$$

④⑨ $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x \Big|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

⑤② $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \int_0^1 \frac{dx}{2\sqrt{1-(x/2)^2}} = \int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u \Big|_0^{1/2}$

$$\begin{aligned} u &= x/2 \\ du &= dx/2 \end{aligned}$$
$$\begin{aligned} &= \sin^{-1}1/2 - \sin^{-1}0 \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6} \end{aligned}$$

8.1

$$\begin{aligned}\textcircled{1} \int e^{2-x} dx &= \int (e^u) (-du) = -e^u + C \\ u &= 2-x \\ du &= -dx \\ &= -e^{2-x} + C\end{aligned}$$

$$\begin{aligned}\textcircled{2} \int \cos\left(\frac{2}{3}x\right) dx &= \int (\cos u) \left(\frac{3}{2} du\right) = \frac{3}{2} \sin u + C \\ u &= \frac{2}{3}x \\ du &= \frac{2}{3} dx \\ &= \frac{3}{2} \sin\left(\frac{2}{3}x\right) + C\end{aligned}$$

$$\begin{aligned}\textcircled{3} \int_0^1 \sin(\pi x) dx &= \int_0^\pi (\sin u) \left(\frac{1}{\pi} du\right) = -\frac{1}{\pi} \cos u \Big|_0^\pi \\ u &= \pi x \\ du &= \pi dx \\ &= -\frac{1}{\pi} (-1 - 1) \\ &= \frac{2}{\pi}\end{aligned}$$

$$\begin{aligned}\textcircled{4} \int \frac{dx}{5^x} &= \int e^{-x \log 5} dx = \int (e^u) \left(-\frac{1}{\log 5} du\right) \\ u &= -x \log 5 \\ du &= (-\log 5) dx \\ &= -\frac{1}{\log 5} e^u + C \\ &= -\frac{1}{\log 5} e^{-x \log 5} + C \\ &= -\frac{1}{\log 5} \cdot \frac{1}{5^x} + C\end{aligned}$$

$$\textcircled{8} \int_0^1 \frac{x^3 dx}{1+x^4} = \int_1^2 \frac{\frac{1}{4} du}{u} = \left[\frac{1}{4} \log |u| \right]_1^2 = \frac{1}{4} (\log 2 - \log 1)$$

$$u = 1+x^4 \qquad \qquad \qquad = \frac{1}{4} \log 2$$

$$du = 4x^3 dx$$

$$\textcircled{9} \int \frac{x dx}{\sqrt{1-x^2}} = \int \frac{-\frac{1}{2} du}{u^{1/2}} = -u^{1/2} + C = -\sqrt{1-x^2} + C$$

$$u = 1-x^2$$

$$du = -2x dx$$

$$\textcircled{11} \int_{-\pi/4}^{\pi/4} \frac{\sin x dx}{\cos^2 x} = \int_{1/\sqrt{2}}^{1/\sqrt{2}} \frac{-du}{u^2} = \left[u^{-1} \right]_{1/\sqrt{2}}^{1/\sqrt{2}} = 0$$

$$u = \cos x$$

$$du = -\sin x dx$$

Also:

$$\int_{-\pi/4}^{\pi/4} \frac{\sin x dx}{\cos^2 x} = \int_{-\pi/4}^{\pi/4} \tan x \sec x dx = \left[\sec x \right]_{-\pi/4}^{\pi/4}$$

$$= \sec(\pi/4) - \sec(-\pi/4)$$

$$= \sqrt{2} - \sqrt{2} = 0$$

$$\textcircled{12} \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int (e^u)(2 du) = 2e^u + C$$

$$u = x^{1/2} \quad = 2e^{\sqrt{x}} + C$$

$$du = \frac{1}{2} x^{-1/2} dx$$

$$\textcircled{15} \int_0^c \frac{dx}{x^2 + c^2} = \int_0^c \frac{dx}{c^2((x/c)^2 + 1)} = \int_0^1 \frac{1}{c} \cdot \frac{du}{u^2 + 1}$$

Note: I'm assuming that $c \neq 0$. If it is indeed that $c = 0$, then $\int_0^c \frac{dx}{x^2 + c^2} = 0$.

$$u = x/c \quad = \left[\frac{\tan^{-1} u}{c} \right]_0^1$$

$$du = dx/c$$

$$= \frac{1}{c} (\tan^{-1} 1 - \tan^{-1} 0)$$

$$= \frac{1}{c} \cdot \frac{\pi}{4}$$

$$\textcircled{22} \int \frac{\log x}{x} dx = \int u du = \frac{1}{2} u^2 + C$$

$$u = \log x \quad = \frac{1}{2} (\log x)^2 + C$$

$$du = \frac{1}{x} dx$$

$$\textcircled{27} \int x \sin(x^2) dx = \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos u + C$$

$$u = x^2 \quad = -\frac{1}{2} \cos(x^2) + C$$

$$du = 2x dx$$

$$\frac{1}{2} du = x dx$$

8.2

$$\textcircled{1} \int \underline{x e^{-x}} dx = (x)(-e^{-x}) - \int (-e^{-x})(dx)$$

$$\underline{u} = x \rightarrow du = dx \quad (\text{simpler})$$

$$v = -e^{-x} \leftarrow \underline{dv} = e^{-x} dx \quad (\text{easy to integrate})$$

$$= -x e^{-x} + \int e^{-x} dx$$

$$= -x e^{-x} - e^{-x} + C$$

If we had picked $u = e^{-x}$ & $dv = x dx$,
then $du = -e^{-x} dx$ & $v = \frac{x^2}{2}$ so that

$$\int x e^{-x} dx = \frac{x^2}{2} e^{-x} - \underbrace{\int \left(\frac{x^2}{2}\right) (-e^{-x} dx)}$$

This is a true statement, but this is now
a harder problem to deal with

If we had picked $u = x e^{-x}$ & $dv = dx$,
then $du = (e^{-x} - x e^{-x}) dx$ & $v = x$, but
again the new integral problem will be
more difficult to deal with.

Lesson: If one division into parts doesn't work,
another might. (Try, try again.)

④ I'm thinking there might be a typo in the problem. I'll do it two ways.

AS WRITTEN:

$$\int x \log(x^2) dx = \int \frac{1}{2} \log u du$$

$$u = x^2$$

$$du = 2x dx$$

$$= \frac{1}{2} (u \log u - u) + C$$

$$= \frac{1}{2} (x^2 \log(x^2) - x^2) + C$$

OR

Recognizing that $\log(x^2) = 2 \log(|x|)$

$$\int \underbrace{2x} du \underbrace{\log(|x|)} = x^2 \log(|x|) - \int x^2 \cdot \frac{1}{x} dx$$

$$\underline{u} = \log(|x|) \quad du = \frac{1}{x} dx$$

$$v = x^2 \quad dv = 2x dx$$

$$= x^2 \log(|x|) - \int x dx$$

$$= x^2 \log(|x|) - \frac{x^2}{2} + D$$

(The same answer!)

④ (continued)

OR

$$\int \underbrace{x \log(x^2)}_u \underbrace{dx}_v = \frac{x^2}{2} \log(x^2) - \int \frac{x^2}{2} \cdot \frac{1}{x^2} 2x dx$$

$$\underline{u} = \log(x^2) \quad du = \frac{1}{x^2} \cdot 2x dx$$

$$\underline{v} = \frac{x^2}{2} \quad \underline{dv} = x dx$$

$$= \frac{x^2}{2} \log(x^2) - \int x dx$$

$$= \frac{x^2}{2} \log(x^2) - \frac{x^2}{2} + E$$

(The same answer!)

(In these last two, choosing $u = x \log(x^2)$ & $dv = dx$ would lead to success through integration by parts, too.)

A DIFFERENT PROBLEM:

$$\int \underbrace{x (\log x)^2}_u \underbrace{dx}_v = \frac{x^2}{2} (\log x)^2 - \int \left(\frac{x^2}{2} \right) \left(2(\log x) \frac{1}{x} dx \right)$$

$$\underline{u} = (\log x)^2 \quad du = 2(\log x) \frac{1}{x} dx$$

$$\underline{v} = \frac{x^2}{2} \quad \underline{dv} = x dx$$

$$= \frac{x^2}{2} (\log x)^2 - \int x \log x dx$$

④ (continued)

We had

$$\int x(\log x)^2 dx = \frac{x^2}{2}(\log x)^2 - \int \frac{x^2}{2} \log x \frac{dx}{x}$$

$$u = \log x \quad du = \frac{1}{x} dx$$

$$v = \frac{x^2}{2} \quad dv = x dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2} \log x - \int \frac{x^2}{2} \frac{1}{x} dx \right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2} \log x + \int \frac{1}{2} x dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + C$$

Other choices for u & dv above would have worked as well.

$$\textcircled{5} \int_0^1 x^2 e^{-x} dx = -x^2 e^{-x} \Big|_0^1 - \int_0^1 (-e^{-x})(2x dx)$$

$$u = x^2 \quad du = 2x dx$$

$$v = -e^{-x} \quad dv = e^{-x} dx$$

$$= -x^2 e^{-x} \Big|_0^1 + \int_0^1 2x e^{-x} dx$$

$$u = 2x \quad du = 2 dx$$

$$v = -e^{-x} \quad dv = e^{-x} dx$$

$$= -x^2 e^{-x} \Big|_0^1 + (2x)(-e^{-x}) \Big|_0^1 - \int_0^1 (-e^{-x})(2 dx)$$

⑤ (continued)

We had:

$$\begin{aligned}\int_0^1 x^2 e^{-x} dx &= \left[-x^2 e^{-x} \right]_0^1 + \left[-2xe^{-x} \right]_0^1 + \int_0^1 2e^{-x} dx \\&= \left[-x^2 e^{-x} \right]_0^1 + \left[-2xe^{-x} \right]_0^1 + \left[-2e^{-x} \right]_0^1 \\&= (-e^{-1} - 0) + (-2e^{-1} - 0) \\&\quad + (-2e^{-1} - (-2)) \\&= 2 - 5e^{-1}\end{aligned}$$

⑨ This is like ④. There are several approaches.

Here is one:

$$\int_1^{e^2} x \log \sqrt{x} dx = \int_1^{e^2} x \cdot \frac{1}{2} \log x \cdot dx$$

$$u = \log x \quad du = \frac{1}{x} dx$$

$$v = \frac{x^2}{4} \quad dv = \frac{1}{2} x dx$$

$$= \left[\frac{x^2}{4} \log x \right]_1^{e^2} - \int_1^{e^2} \frac{x^2}{4} \cdot \frac{1}{x} dx$$

$$= \left[\frac{x^2}{4} \log x \right]_1^{e^2} - \left[\frac{x^2}{8} \right]_1^{e^2}$$

$$= \left(\frac{1}{4} (e^4) (2) - 0 \right) - \left(\frac{1}{8} e^4 - \frac{1}{8} \right)$$

$$= \frac{3}{8} e^4 + \frac{1}{8}$$

8.3

$$\cos^2 x + \sin^2 x = 1$$

$$\textcircled{1} \int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

$$u = \cos x$$

$$du = -\sin x$$

$$= \int (1 - u^2)(-du)$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{1}{3} \cos^3 x - \cos x + C$$

$$\textcircled{2} \int \sin^3 x \cos^2 x \, dx = \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$= \int (1 - u^2)(u^2)(-du)$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$$

$$\textcircled{5} \int \sin^3 x \cos^3 x \, dx = \int \sin x (1 - \cos^2 x) \cos^3 x \, dx$$

$$\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned} \quad = \int (1 - u^2)(u^3)(-du)$$

$$= \frac{1}{6} u^6 - \frac{1}{4} u^4 + C$$

$$= \frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + C$$

OR

$$\int \sin^3 x \cos^3 x \, dx = \int (\sin^2 x)(1 - \sin^2 x) \cos x \, dx$$

$$\begin{aligned} w &= \sin x \\ dw &= \cos x \, dx \end{aligned} \quad = \int w^3 (1 - w^2) \, dw$$

$$= \frac{1}{4} w^4 - \frac{1}{6} w^6 + D$$

$$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + D$$

The same answer!

(If you're skeptical, plot the two answers — with $C=0$ & $D=0$ — and verify that the graphs are just vertical translates of one another (that is, that $\frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x$ and $\frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x$ differ by a constant).)

$$\cos^2 x + \sin^2 x = 1 \quad \text{and} \quad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\text{so that} \quad \cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

$$\text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\begin{aligned} \textcircled{8} \quad \int \sin^2 x \cos^4 x \, dx &= \int (\sin^2 x)(\cos^2 x)^2 \, dx \\ &= \int \left(\frac{1}{2}(1 - \cos(2x)) \right) \left(\frac{1}{2}(1 + \cos(2x)) \right)^2 \, dx \\ &= \int \frac{1}{8} (1 - \cos(2x)) (1 + 2\cos(2x) + \cos^2(2x)) \, dx \\ &= \int \frac{1}{8} (1 + 2\cos(2x) + \cos^2(2x) - \cos(2x) \\ &\quad - 2\cos^2(2x) - \cos^3(2x)) \, dx \\ &= \int \frac{1}{8} (1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)) \, dx \\ &= \int \frac{1}{8} \, dx + \int \frac{1}{8} \cos(2x) \, dx - \int \frac{1}{8} \cos^2(2x) \, dx - \int \frac{1}{8} \cos^3(2x) \, dx \end{aligned}$$

$$\int \frac{1}{8} \, dx = \frac{1}{8} x + C_1$$

$$\begin{aligned} \int \frac{1}{8} \cos(2x) \, dx &= \int \frac{1}{16} \cos u \, du = \frac{1}{16} \sin u + C_2 \\ \text{Let } u &= 2x \\ du &= 2 \, dx &= \frac{1}{16} \sin(2x) + C_2 \end{aligned}$$

⑧ (continued)

$$\int \frac{1}{8} \cos^2(2x) dx = \int \frac{1}{8} \left(\frac{1}{2} (1 + \cos(4x)) \right) dx$$

$$u = 4x$$

$$du = 4 dx$$

$$= \int \left(\frac{1}{16} + \frac{1}{16} \cos(4x) \right) dx$$

$$= \int \left(\frac{1}{16} + \frac{1}{16} \cos u \right) \left(\frac{1}{4} du \right)$$

$$= \frac{1}{4} \left(\frac{1}{16} u + \frac{1}{16} \sin u \right) + C_3$$

$$= \frac{1}{16} x + \frac{1}{64} \sin(4x) + C_3$$

$$\int \frac{1}{8} \cos^3(2x) dx = \int \frac{1}{8} \cos^2(2x) \cos(2x) dx$$

$$= \int \frac{1}{8} (1 - \sin^2(2x)) \cos(2x) dx$$

$$u = \sin(2x)$$

$$du = 2 \cos(2x) dx = \int \frac{1}{8} (1 - u^2) \left(\frac{1}{2} du \right)$$

$$= \frac{1}{16} u - \frac{1}{48} u^3 + C_4$$

$$= \frac{1}{16} \sin(2x) - \frac{1}{48} \sin^3(2x) + C_4$$

⑧ (continued)

$$\begin{aligned}\text{Thus } \int \sin^2 x \cos^4 x dx &= \left(\frac{1}{8} x \right) + \left(\frac{1}{16} \sin(2x) \right) \\ &\quad - \left(\frac{1}{16} x + \frac{1}{64} \sin(4x) \right) - \left(\frac{1}{16} \sin(2x) - \frac{1}{48} \sin^3(2x) \right) + C \\ &= \frac{1}{16} x - \frac{1}{64} \sin(4x) + \frac{1}{48} \sin^3(2x) + C\end{aligned}$$

There is more than one way to do this problem. You can check your solution against this one by verifying it differs only by a constant over all x .

$$\begin{aligned}\textcircled{13} \int_0^\pi \sin^4 x dx &= \int_0^\pi \left(\frac{1}{2}(1 - \cos(2x)) \right)^2 dx \\ &= \int_0^\pi \frac{1}{4} (1 - 2\cos(2x) + \cos^2(2x)) dx \\ &= \int_0^\pi \frac{1}{4} dx + \int_0^\pi -\frac{1}{2} \cos(2x) dx + \int_0^\pi \frac{1}{4} \cos^2(2x) dx \\ \int_0^\pi \frac{1}{4} dx &= \frac{1}{4} x \Big|_0^\pi = \frac{1}{4} \pi - 0 = \frac{1}{4} \pi \\ \int_0^\pi -\frac{1}{2} \cos(2x) dx &= -\frac{1}{4} \sin(2x) \Big|_0^\pi = 0 - 0 = 0\end{aligned}$$

(13) (continued)

$$\begin{aligned}\int_0^{\pi} \frac{1}{4} \cos^2(2x) dx &= \int_0^{\pi} \frac{1}{8} (1 + \cos(4x)) dx \\&= \left[\frac{1}{8} x + \frac{1}{32} \sin(4x) \right]_0^{\pi} \\&= \left(\frac{1}{8} \pi + 0 \right) - (0 + 0) \\&= \frac{1}{8} \pi\end{aligned}$$

Thus

$$\int_0^{\pi} \sin^4 x dx = \frac{1}{4} \pi + 0 + \frac{1}{8} \pi = \frac{3}{8} \pi$$