

5.6

$$\textcircled{1} \int \frac{dx}{(2-3x)^2}$$

by reverse
power rule

We would know how to integrate x^{-2} but we have $(2-3x)^{-2}$. This leads us to the substitution:

$$u = 2 - 3x$$

So that

$$du = -3dx \quad \text{or} \quad -\frac{1}{3}du = dx$$

Thus

$$\begin{aligned} \int \frac{dx}{(2-3x)^2} &= \int \frac{-\frac{1}{3}du}{u^2} = \int -\frac{1}{3}u^{-2} du \\ &= \left(-\frac{1}{3}\right) \left(\frac{1}{-1}u^{-1}\right) + C \\ &= \frac{1}{3} \cdot \frac{1}{2-3x} + C \end{aligned}$$

$$\textcircled{4} \int \sqrt{ax+b} dx$$

We would know how to integrate $x^{1/2}$, but we're stuck with $(ax+b)^{1/2}$. This leads to

$$u = ax + b$$

$$du = a dx \quad \longrightarrow \quad \frac{1}{a} du = dx$$

$$\int \sqrt{ax+b} dx = \int (\sqrt{u}) \left(\frac{1}{a} du\right)$$

$$= \int \frac{1}{a} u^{1/2} du$$

$$= \left(\frac{1}{a}\right) \left(\frac{2}{3} u^{3/2}\right) + C$$

$$= \frac{2}{3a} (ax+b)^{3/2} + C$$

Note: This presumes $a \neq 0$. If indeed $a=0$, this is a very simple problem and doesn't need a substitution

$$\textcircled{5} \int (ax+b)^{3/4} dx$$

This follows as in $\textcircled{4}$. Letting:

$$u = ax+b$$

$$du = a dx$$

$$\rightarrow \frac{1}{a} du = dx$$

$$\int (ax+b)^{3/4} dx = \int (u^{3/4}) \left(\frac{1}{a} du \right)$$

$$= \left(\frac{1}{a} \right) \left(\frac{4}{7} u^{7/4} \right) + C$$

$$= \frac{4}{7a} (ax+b)^{7/4} + C$$

Again, if $a=0$, this is a much simpler problem.

$$\textcircled{7} \int \frac{t}{(4t^2+9)^2} dt$$

We don't like $(4t^2+9)^{-2}$; we'd prefer t^{-2} instead. This suggests:

$$u = 4t^2+9$$

$$du = 8t dt$$

Before it was easy to express dt in terms of du alone (all we had to do was divide by a constant). As luck would have it, though, $t dt$ appears in the integrand. Thus

$$\int \frac{t}{(4t^2+9)^2} dt = \int \frac{1/8 du}{u^2}$$

$$= \left(\frac{1}{8} \right) \left(\frac{1}{-1} u^{-1} \right) + C$$

$$= -\frac{1}{8} \cdot \frac{1}{4t^2+9} + C$$

$$\textcircled{11} \int \frac{s}{(1+s^2)^3} ds$$

The same applies.

$$u = 1+s^2$$

$$du = 2s ds$$

$$\int \frac{s}{(1+s^2)^3} ds = \int \frac{1/2 du}{u^3} = \left(\frac{1}{2} \right) \left(\frac{1}{-2} u^{-2} \right) + C$$

$$= -\frac{1}{4} \cdot \frac{1}{(1+s^2)^2} + C$$

$$(12) \int \frac{2s}{\sqrt[3]{6-5s^2}} ds$$

$$u = 6 - 5s^2$$

$$du = -10s ds \rightarrow -\frac{1}{5} du = 2s ds$$

$$\begin{aligned} \int \frac{2s}{\sqrt[3]{6-5s^2}} ds &= \int \frac{-\frac{1}{5} du}{\sqrt[3]{u}} \\ &= \int -\frac{1}{5} u^{-1/3} du \\ &= \left(-\frac{1}{5}\right) \left(\frac{3}{2} u^{2/3}\right) + C \\ &= -\frac{3}{10} (6-5s^2)^{2/3} + C \end{aligned}$$

$$(15) \int 5x(x^2+1)^{-3} dx$$

$$\text{Let } u = x^2 + 1$$

$$\text{so that } du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$\begin{aligned} \int 5(x^2+1)^{-3} x dx &= \int (5u^{-3}) \left(\frac{1}{2} du\right) \\ &= \left(\frac{5}{2}\right) \left(\frac{1}{-2} u^{-2}\right) + C \\ &= -\frac{5}{4} (x^2+1)^{-2} + C \end{aligned}$$

$$(16) \int 2x^3(1-x^4)^{-1/4} dx$$

$$\text{Let } u = 1 - x^4$$

$$\text{so that } du = -4x^3 dx \rightarrow -\frac{1}{4} du = x^3 dx$$

$$\begin{aligned} \int 2(1-x^4)^{-1/4} x^3 dx &= \int (2u^{-1/4}) \left(-\frac{1}{4} du\right) \\ &= \int -\frac{1}{2} u^{-1/4} du \\ &= \left(-\frac{1}{2}\right) \left(\frac{4}{3} u^{3/4}\right) + C \\ &= -\frac{2}{3} (1-x^4)^{3/4} + C \end{aligned}$$

$$\textcircled{21} \int_0^1 x(x^2+1)^3 dx$$

$$\text{Let } u = x^2 + 1$$

$$1 = (0)^2 + 1$$

$$2 = (1)^2 + 1$$

$$\text{so that } du = 2x dx$$

$$\int_{x=0}^{x=1} x(x^2+1)^3 dx = \int_{u=1}^{u=2} \frac{1}{2} du = x dx$$

$$= \left(\frac{1}{2} \right) \left(\frac{1}{4} u^4 \right) \Big|_1^2$$

$$= \frac{1}{8} 2^4 - \frac{1}{8} 1^4 = 17/8$$

Alternatively, you can leave the limits in x and back substitute after integrating:

$$\int_{x=0}^{x=1} x(x^2+1)^3 dx = \int_{x=0}^{x=1} \frac{1}{2} u^3 du$$

$$= \left(\frac{1}{2} \right) \left(\frac{1}{4} u^4 \right) \Big|_{x=0}^{x=1}$$

$$= \frac{1}{8} (x^2+1)^4 \Big|_0^1$$

$$= \frac{1}{8} (1^2+1)^4 - \frac{1}{8} (0^2+1)^4$$

$$= 17/8$$

By changing the limits when changing variables, all we do is rearrange some of the work. (That is, evaluating 1^2+1 and 0^2+1 comes earlier rather than later.) But doing it immediately (as in the top) rather than later (as in the bottom) helps avoid confusion. People assume that limits are given in terms of the variable against which you are integrating; this allows you to leave off the " $x=$ " and " $u=$ " designation.

$$\textcircled{22} \int_{-1}^0 3x^2 (4 + 2x^3)^2 dx$$

$$u = 4 + 2x^3$$

$$du = 6x^2 dx \rightarrow \frac{1}{2} du = 3x^2 dx$$

$$2 = 4 + 2(-1)^3$$

$$4 = 4 + 2(0)^3$$

$$\int_{-1}^0 3x^2 (4 + 2x^3)^2 dx = \int_2^4 (u^2) \left(\frac{1}{2} du \right)$$

$$= \frac{1}{6} u^3 \Big|_2^4$$

$$= \frac{1}{6} 4^3 - \frac{1}{6} 2^3$$

$$= 28/3$$

$$\textcircled{39} \int \cos(3x+1) dx$$

$$u = 3x+1$$

$$du = 3dx \rightarrow \frac{1}{3} du = dx$$

$$\int \cos(3x+1) dx = \int (\cos u) \left(\frac{1}{3} du \right)$$

$$= \frac{1}{3} \sin u + C$$

$$= \frac{1}{3} \sin(3x+1) + C$$

$$\textcircled{40} \int \sin(2\pi x) dx = \int (\sin u) \left(\frac{1}{2\pi} du \right)$$

$$u = 2\pi x$$

$$du = 2\pi dx = -\frac{1}{2\pi} \cos u + C$$

$$= -\frac{1}{2\pi} \cos(2\pi x) + C$$

$$\textcircled{44} \int \sin^2 x \cos x dx = \int u^2 du$$

$$u = \sin x$$

$$du = \cos x dx$$

$$= \frac{1}{3} u^3 + C$$

$$= \frac{1}{3} \sin^3 x + C$$

$$\textcircled{52} \int (1 + \tan^2 x) \sec^2 x dx = \int (1 + u^2) du$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

$$= u + \frac{1}{3} u^3 + C$$

$$= \tan x + \frac{1}{3} \tan^3 x + C$$

$$\textcircled{55} \int \frac{\sec^2 x}{\sqrt{1 + \tan x}} dx = \int \frac{du}{u^{1/2}}$$

$$u = 1 + \tan x$$

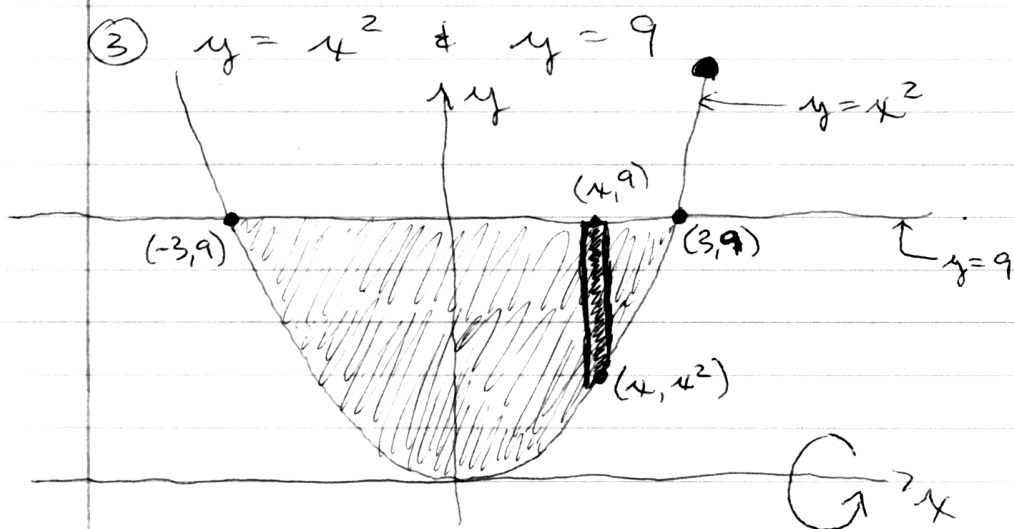
$$du = \sec^2 x dx$$

$$= 2 u^{1/2} + C$$

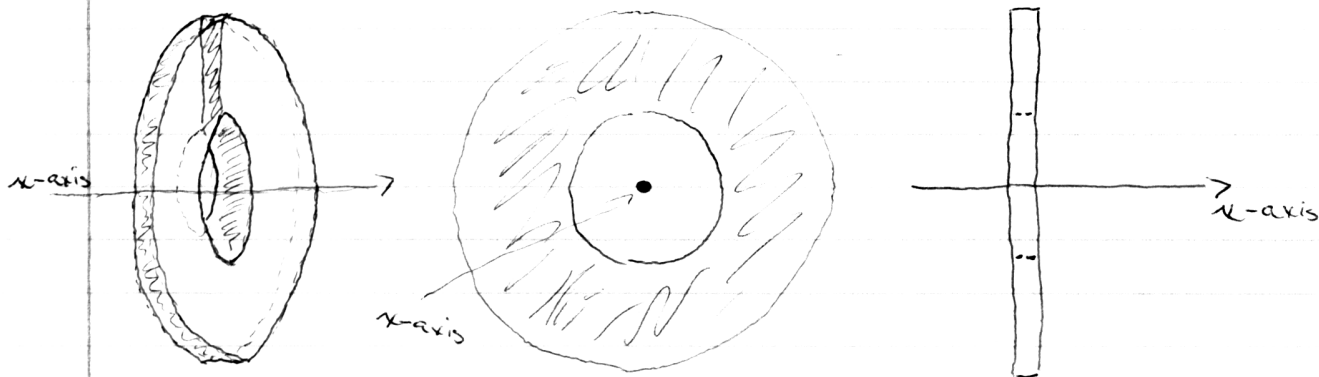
$$= 2(1 + \tan x)^{1/2} + C$$

(3, 11, 15)

6.2 In the exercises below, sketch the region bounded by the curves and find the volume of the solid generated by revolving this region about the x -axis.



Suppose we take the darkly shaded area above and rotate it around the x -axis. The solid swept out looks like a washer (pancake with a hole cut out, a vinyl 45, et cetera).



$$\begin{aligned}
 \text{volume of washer} &= (\text{area of top face of washer}) \times (\text{thickness of washer}) \\
 &= \underbrace{(\pi \cdot 9^2 - \pi(x^2)^2)}_{\text{area of an annulus}} \cdot \underbrace{dx}_{\text{a teeny change in } x}
 \end{aligned}$$

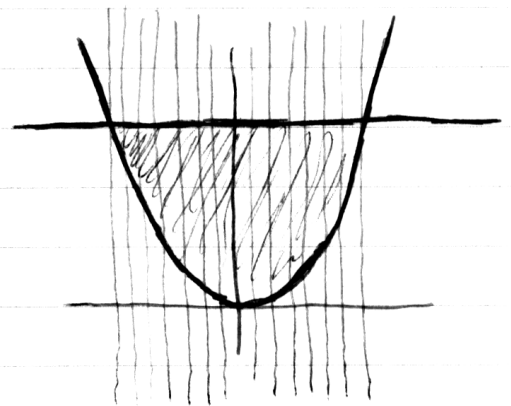
③ (cont)

(in general)
We now know what a small piece of the volume of the solid looks like. To find the total volume, we need to add up many similar such little volumes:

$$\begin{aligned}\text{total volume} &= \int_{x=-3}^{x=3} (\pi 9^2 - \pi (x^2)^2) dx \\ &= \int_{-3}^3 (81\pi - \pi x^4) dx \\ &= \left[81\pi x - \frac{\pi}{5} x^5 \right]_{-3}^3 \\ &= \left(81\pi \cdot 3 - \frac{\pi}{5} \cdot 3^5 \right) - \left(81\pi(-3) - \frac{\pi}{5}(-3)^5 \right) \\ &= \frac{1944}{5} \pi \quad \left(= (388\frac{4}{5})\pi \right)\end{aligned}$$

Two questions linger: why consider the darkly shaded area (and what it does when revolved) in the first place, and why are the limits of integration as they are above?

If we had simply wanted to find the area of the shaded region to the right, we could use integration by splitting the region into a bunch of strips, approximating each strip as a rectangle with area $(9-x^2)dx$ and adding up all those areas to get

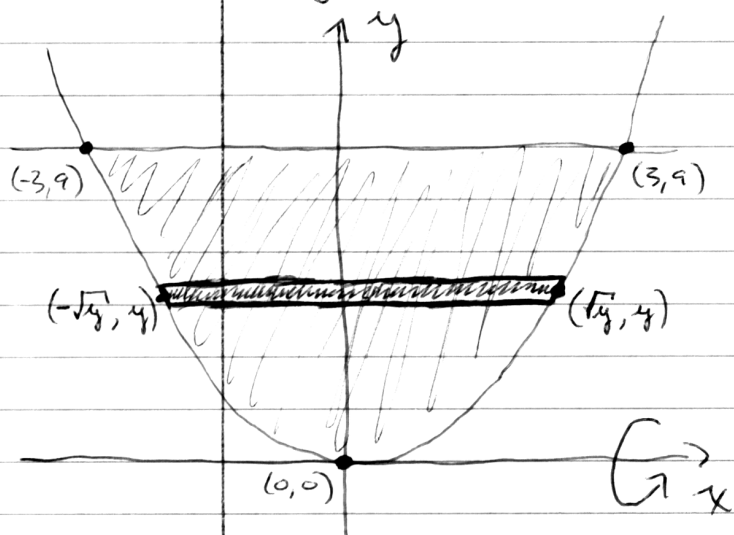


③ (cont) $\int_{-3}^3 (9-x^2) dx = 36$. In finding

the volume of the solid generated by revolving the region about the x -axis, it makes sense to split the region into the same small strips, approximate them as rectangles, and find the volumes generated by those rectangles as they are revolved about the x -axis.

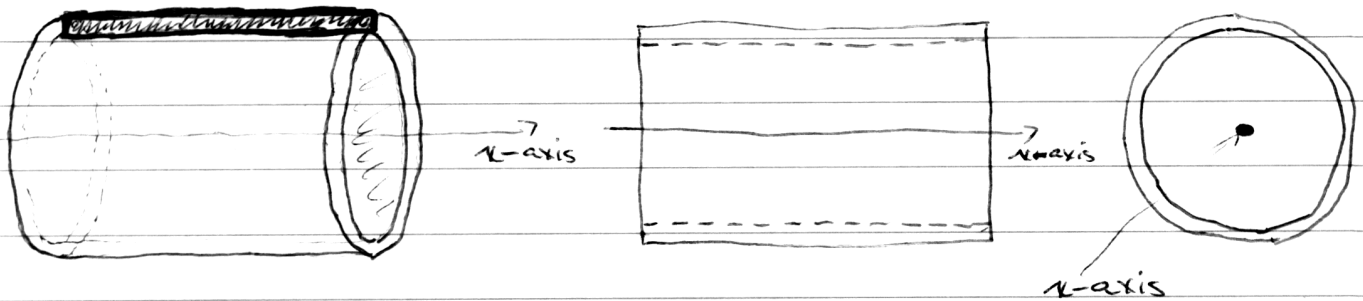
Why start integrating at $x=-3$ and stop at $x=+3$? For the same reasons we did in finding the area of the region itself: these are where the strips start and stop. (See §5.4 and §6.1.)

Let's do this problem a second way (since it's coming up anyway and we also need the practice): instead of slicing the original region into strips perpendicular to the x -axis (the axis of rotation), slice the region into strips parallel to the x -axis.



As we rotate this strip about the x -axis, it sweeps out a solid that looks like a thin, cylindrical shell (picture a soup can with its top and bottom removed). If we can find the (a piece of volume of this shell, the region)

③ (cont)



(and all the ones like it), we can find the total volume of the solid swept out by the whole region. Note we want to find the volume of the shell ^{itself}, and not the volume contained by the shell (the volume of the metal used to make the soup can, and not the volume of soup that the can would hold).

We can find this volume by cutting the shell along its length, unrolling it into a box shape, and finding the volume of the resulting box (LWH).

$$\begin{aligned}
 \text{volume of shell} &= (\text{thickness of shell}) \times \\
 &\quad (\text{length of shell}) \times (\text{width of unrolled box}) \\
 &= (\text{thickness}) \times (\text{length}) \times (\text{circumference of shell}) \\
 &= (\text{thickness}) \times (\text{length}) \times (2\pi \times \text{radius of shell}) \\
 &= \underbrace{(dy)}_{\substack{\text{a teeny} \\ \text{change in } y}} \times \underbrace{(\sqrt{y} - (-\sqrt{y}))}_{\substack{\text{right} \\ \text{endpt} - \text{left} \\ \text{endpt} \\ \text{(of the strip)}}} \times \underbrace{(2\pi \times y)}_{\substack{\text{distance of the} \\ \text{strip from the} \\ \text{x-axis}}} \\
 &= (dy)(2\sqrt{y})(2\pi y) \\
 &= 4\pi y^{3/2} dy
 \end{aligned}$$

Thus:

$$\text{total volume} = \int_{y=0}^{y=9} 4\pi y^{3/2} dy$$

(Again, the limits of integration come from the

③ (cont) extents of the region not the extents of the solid (note that: area of region = $\int_0^9 (\sqrt{y} - (-\sqrt{y})) dy = 36$).

$$\text{total volume} = \int_0^9 4\pi y^{3/2} dy$$

$$= \left(4\pi \left(\frac{2}{5} y^{5/2} \right) \right) \Big|_0^9$$

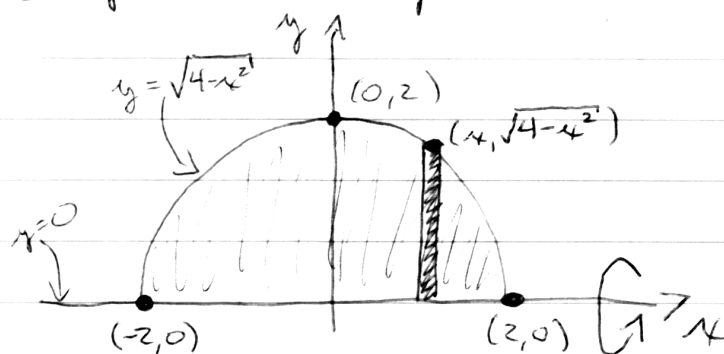
$$= \left(4\pi \left(\frac{2}{5} 9^{5/2} \right) \right) - \left(4\pi \left(\frac{2}{5} 0^{5/2} \right) \right)$$

$$= 4\pi \cdot \frac{2}{5} \cdot 243 = \frac{1944}{5} \pi \quad (= (388 \frac{4}{5}) \pi)$$

is the
This same answer as we got with the other method. (Yah! We didn't make any mistakes.)

If you are given a choice, use the method you are most comfortable with. However, it will usually be the case that one method will be more tractable than the other (and it won't always be the disc/washer method; in this problem it just happened to be that way). Of course, sometimes you're not given a choice...

⑪ $y = \sqrt{4-x^2}$ & $y=0$



Let's first find the volume by considering the contributions to the volume from ^{revolving} strips of area gotten by cutting up the shaded region into strips perpendicular to

The x -axis (the axis of rotation). These bits of volume are going to be discs (pancakes, half-dollars, whatever). The volume of any one disc will be:

(ii) (cont) vol of disc = (area of top face) \times (thickness of disc)

$$= (\pi(\sqrt{4-x^2}-0)^2) \times (dx)$$

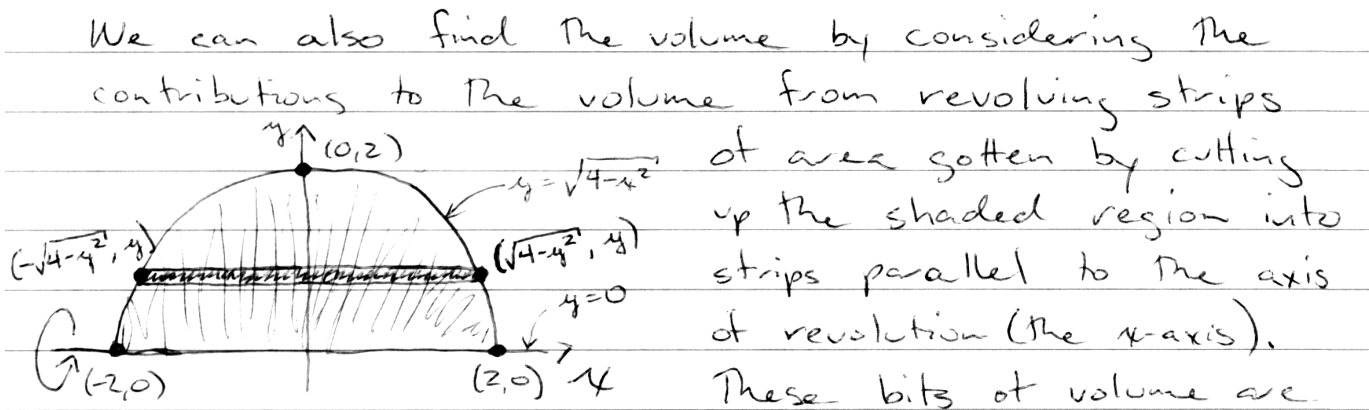
$$= \pi(4-x^2)dx$$

total volume = $\int_{-2}^2 \pi(4-x^2)dx$

$$= \pi \left(4x - \frac{x^3}{3} \right) \Big|_{-2}^2$$

$$= \left(\pi \left(4 \cdot 2 - \frac{2^3}{3} \right) \right) - \left(\pi \left(4(-2) - \frac{(-2)^3}{3} \right) \right)$$

$$= \frac{32}{3} \pi$$



going to be shells (top & bottomless soup cans, whatever).

The volume of any one shell will be:

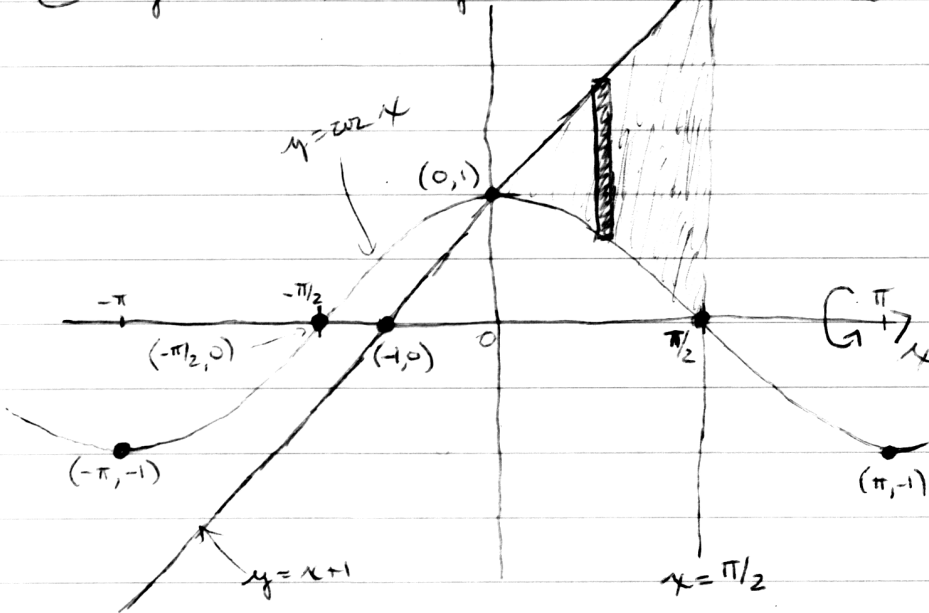
$$\begin{aligned} \text{vol of shell} &= (\text{length of shell unrolled}) \times (\text{width of shell unrolled}) \times (\text{thickness of shell}) \\ &= (\text{length of shell}) \times (\text{circumference of shell}) \times (\text{thickness}) \\ &= (\sqrt{4-y^2} - -\sqrt{4-y^2}) \times (2\pi y) \times (dy) \\ &= 4\pi y \sqrt{4-y^2} dy \end{aligned}$$

$$\begin{aligned} \text{total volume} &= \int_0^2 4\pi y \sqrt{4-y^2} dy \\ &= \int_4^0 -2\pi \sqrt{u} du \end{aligned}$$

$$\begin{aligned} \text{Let } u &= 4-y^2 \\ du &= -2y dy \\ 4 &= 4-(0)^2 \\ 0 &= 4-(2)^2 \end{aligned}$$

$$\begin{aligned}
 \textcircled{11} \text{ (cont) total volume} &= \int_4^0 -2\pi x^{1/2} dx \\
 &= (-2\pi) \left(\frac{2}{3} x^{3/2} \right) \Big|_4^0 \\
 &= \left((-2\pi) \left(\frac{2}{3} 0^{3/2} \right) \right) - \left((-2\pi) \left(\frac{2}{3} 4^{3/2} \right) \right) \\
 &= \frac{32}{3} \pi
 \end{aligned}$$

$$\textcircled{15} \quad y = \cos 2x \text{ \& } y = x+1 \text{ \& } x = \pi/2$$



$$\text{volume of washer} = \left(\underbrace{\pi(x+1)^2}_{\text{outer radius}} - \underbrace{\pi(\cos 2x)^2}_{\text{inner radius}} \right) (dx)$$

$$\text{total volume} = \int_0^{\pi/2} (\pi(x+1)^2 - \pi \cos^2 2x) dx$$

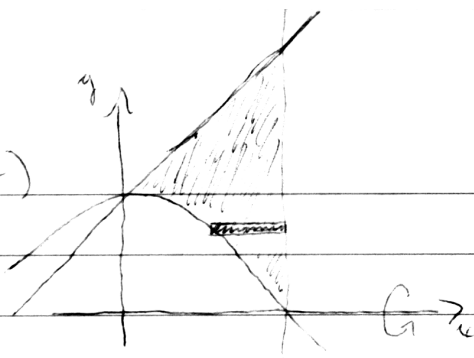
$$\text{Note: } \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$\text{since } \sin^2 x + \cos^2 x = 1, \text{ Thus } \cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\text{total volume} = \int_0^{\pi/2} \left(\pi(x^2 + 2x + 1) - \frac{\pi}{2}(1 + \cos(2x)) \right) dx$$

$$\begin{aligned}
 &= \pi \left(\frac{x^3}{3} + 2x^2 + x \right) - \frac{\pi}{2} \left(x + \frac{1}{2} \sin(2x) \right) \Big|_0^{\pi/2} \\
 &= \pi \left(\frac{\pi^3}{24} + \frac{\pi^2}{4} + \frac{\pi}{2} \right) - \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^4}{24} + \frac{\pi^3}{4} + \frac{\pi^2}{4}
 \end{aligned}$$

15 (cont)



vol of shell = (width) \times (circumference) \times (thickness)

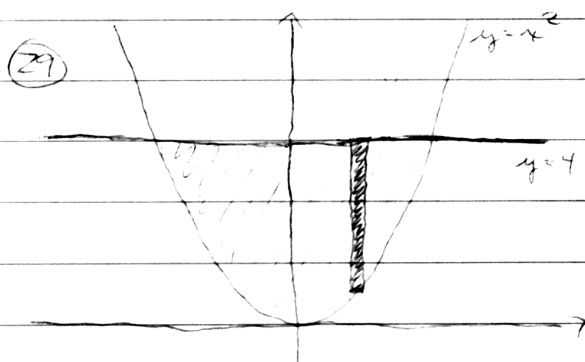
$$= (\text{right} - \text{left}) \times (2\pi y) \times (dy)$$

$$= \begin{cases} (\pi/2 - \cos^{-1} y)(2\pi y)(dy) & \text{for } 0 \leq y \leq 1 \\ (\pi/2 - (y-1))(2\pi y)(dy) & \text{for } 1 \leq y \leq 1 + \pi/2 \end{cases}$$

$$\text{total vol} = \int_0^1 (\pi/2 - \cos^{-1} y)(2\pi y) dy + \int_1^{1+\pi/2} (\pi/2 - (y-1))(2\pi y) dy$$

you'll know how to do this soon

you know how to do this now



② A square has ^{four} sides of equal length. It's area is the length of one of those sides squared. One of the sides lies on the base and has length

$4-x^2$. Thus vol of slice = $(4-x^2)^2 dx$ and

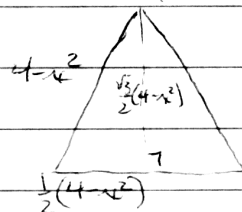
$$\text{total vol} = \int_{-2}^2 (4-x^2)^2 dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{-2}^2$$

$$= \left(32 - \frac{64}{3} + \frac{32}{5} \right) - \left(-32 + \frac{64}{3} - \frac{32}{5} \right)$$

$$= \frac{512}{15}$$

③ Again, the base of the triangle has length $4-x^2$ (and all the sides have the same length. Thus it

has height $\frac{\sqrt{3}}{2}(4-x^2)$ and area $\frac{\sqrt{3}}{4}(4-x^2)^2$

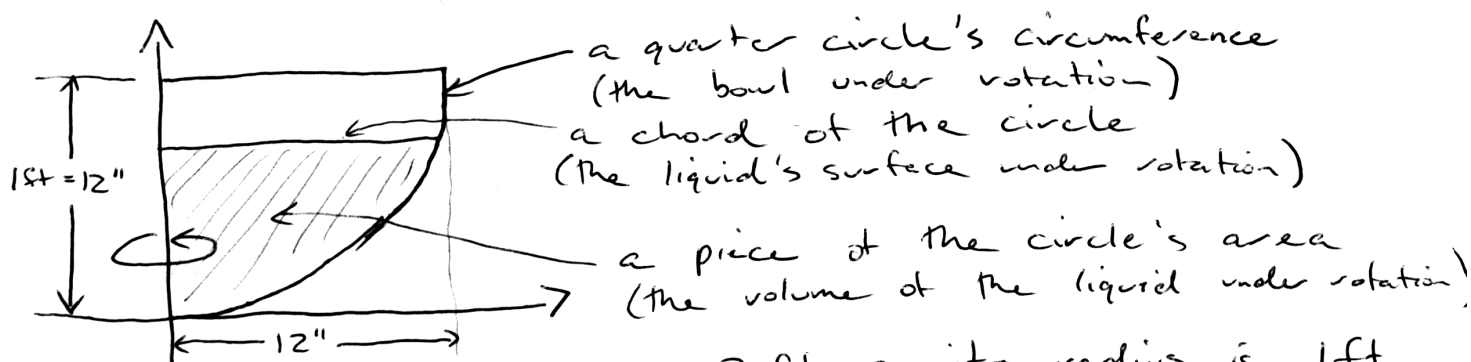


(29) (b) Thus vol of slice = $\frac{\sqrt{3}}{4} (4-x^2)^2 dx$
 and total volume = $\int_{-2}^2 \frac{\sqrt{3}}{4} (4-x^2)^2 dx$

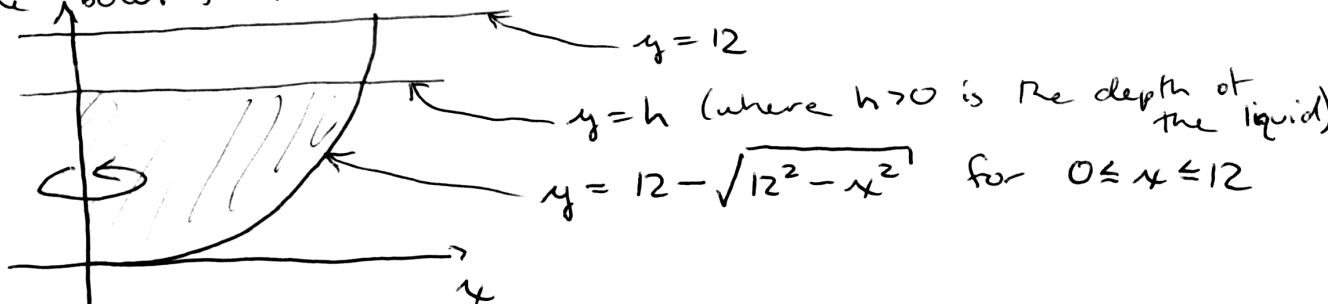
$$= \frac{\sqrt{3}}{4} \left(\frac{512}{15} \right) = \frac{128\sqrt{3}}{15}$$

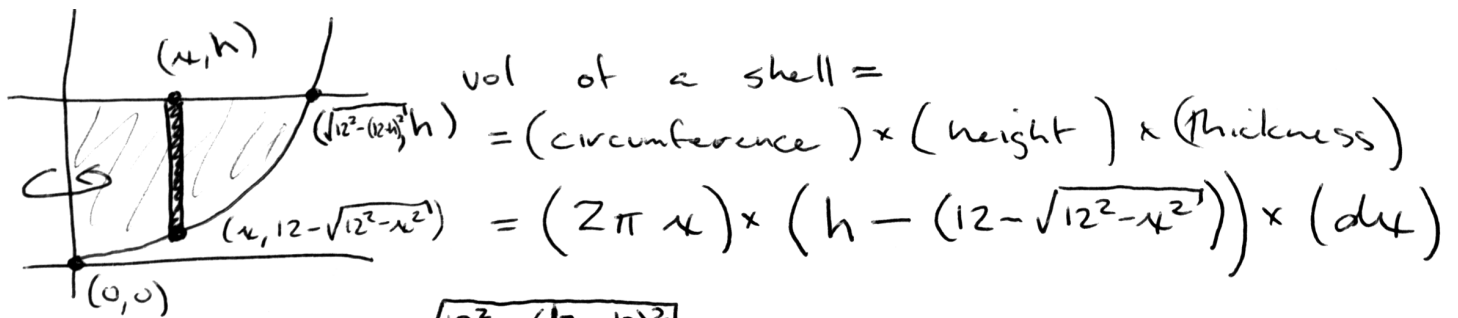
(c) The base of the semicircle has length $4-x^2$
 so it has a radius of $\frac{1}{2}(4-x^2)$ and an
 area of $\frac{1}{2}\pi \left(\frac{1}{2}(4-x^2) \right)^2 = \frac{\pi}{8} (4-x^2)^2$. The
 solid has a volume of $\int_{-2}^2 \frac{\pi}{8} (4-x^2)^2 dx = \frac{\pi}{8} \left(\frac{512}{15} \right) = \frac{64\pi}{15}$

(42) Fortunately for us, a hemispherical bowl
 (and the liquid contained therein) has radial
 symmetry (also known as axial symmetry). In
 other words, we can find a region in the
 plane which, when revolved around an axis,
 generates ^(sweeps out) a solid in the form of the
 liquid in the bowl.



The bowl's diameter is 2 ft so its radius is 1 ft.



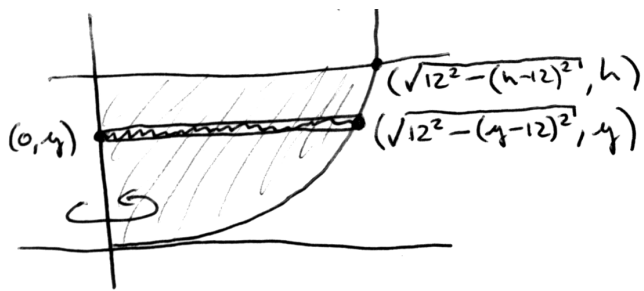


$$\begin{aligned} \text{vol of liquid} &= \int_0^{\sqrt{12^2 - (h-12)^2}} (2\pi x (h - (12 - \sqrt{12^2 - x^2}))) dx \\ &= 2\pi \int_0^{\sqrt{12^2 - (h-12)^2}} (x(h-12) + x\sqrt{12^2 - x^2}) dx \\ &= 2\pi \left[\frac{x^2}{2}(h-12) + \frac{1}{3}(12^2 - x^2)^{3/2} \right]_0^{\sqrt{12^2 - (h-12)^2}} \\ &= 2\pi \left(\frac{(12^2 - (h-12)^2)}{2}(h-12) + \frac{1}{3}((h-12)^{3/2}) \right) \\ &\quad - 2\pi \left(0 - \frac{1}{3}12^3 \right) \\ &= 2\pi \left(\frac{1}{2} \cdot 12^2(h-12) - \frac{1}{6}(h-12)^3 + \frac{1}{3}12^3 \right) \\ &= \frac{2\pi}{6} (-h^3 + 36h^2) = \frac{\pi}{3} h^2 (3 \cdot 12 - h) \end{aligned}$$

$$\begin{aligned} (h-12)^3 &\neq ((h-12)^2)^{3/2} = -(h-12)^3 \\ &= (12-h)^3 \end{aligned}$$

Note: $h < 12$ so $h-12 < 0$

$$\text{Thus } ((h-12)^2)^{1/2} = 12-h$$



$$\text{vol of a disc} = \pi (\text{radius of disc})^2 (\text{thickness})$$

$$= \pi \left(\sqrt{12^2 - (y-12)^2} \right)^2 (dy)$$

$$\text{vol of liquid} = \int_0^h \pi (12^2 - (y-12)^2) dy$$

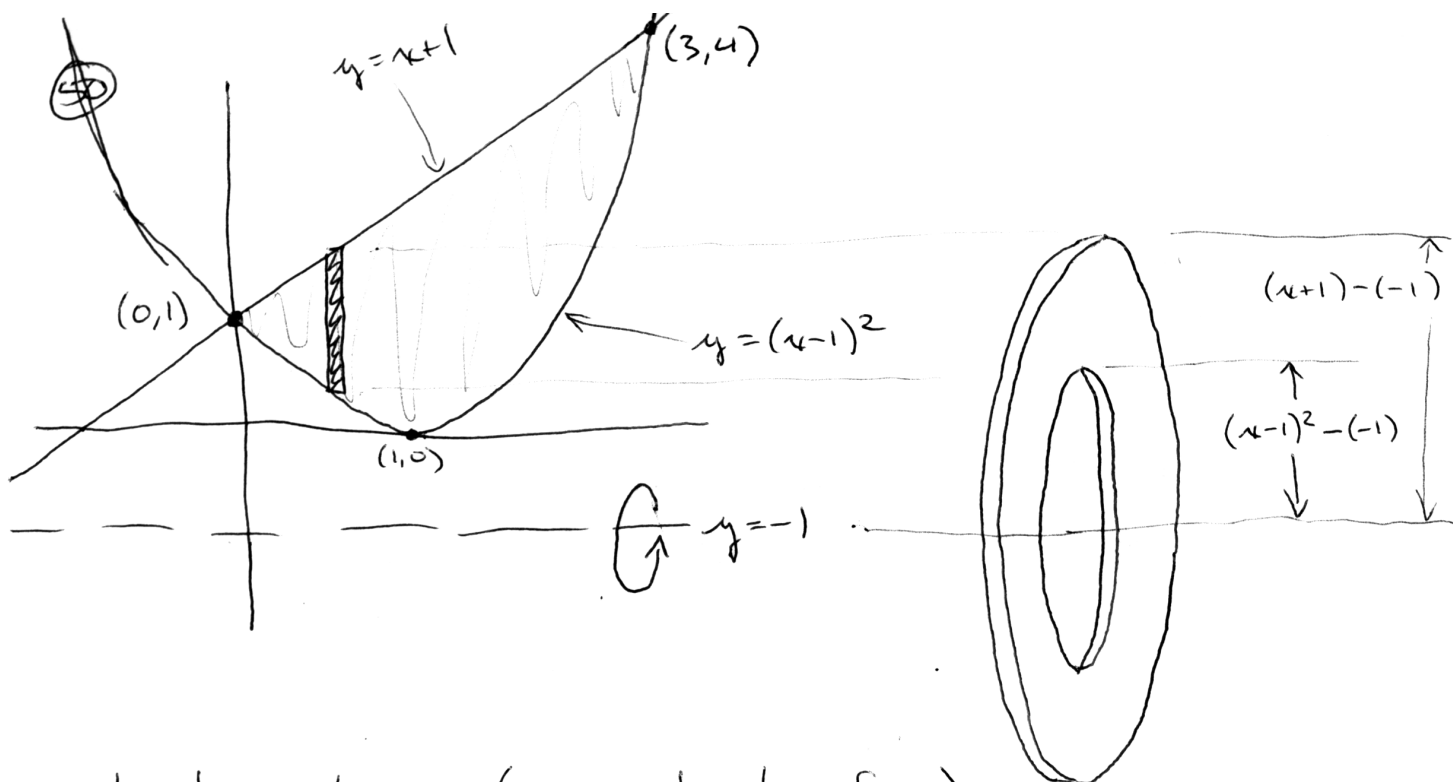
$$= \pi \left[12^2 y - \frac{(y-12)^3}{3} \right]_0^h$$

$$= \pi \left(12^2 h - \frac{1}{3} (h-12)^3 \right) - \pi \left(0 - \frac{1}{3} (-12)^3 \right)$$

$$= \frac{\pi}{3} \left(3 \cdot 12^2 h - \frac{1}{3} (h-12)^3 - 12^3 \right)$$

$$= \frac{\pi}{3} (-h^3 + 36h) = \frac{\pi}{3} \cdot h^2 (3 \cdot 12 - h)$$

When the liquid was 1" from the top, it was $h = 11$ inches deep. Thus there were $\frac{\pi}{3} \cdot 121 \cdot (36 - 11)$ cubic inches to begin with. In the end, the liquid was only $h = 2$ inches deep leaving $\frac{\pi}{3} \cdot 4 \cdot (36 - 4)$ cubic inches. Thus $\frac{\pi}{3} \cdot 121(25) - \frac{\pi}{3} \cdot 4(32) = \frac{\pi}{3} \cdot 2897$ cubic inches were consumed.



vol of washer = (area of top face) \times (thickness)

$$= \left[\pi (\text{outer radius})^2 - \pi (\text{inner radius})^2 \right] \times (\text{thickness})$$

$$= \left[\pi ((x+1) - (-1))^2 - \pi ((x-1)^2 - (-1))^2 \right] \times (dx)$$

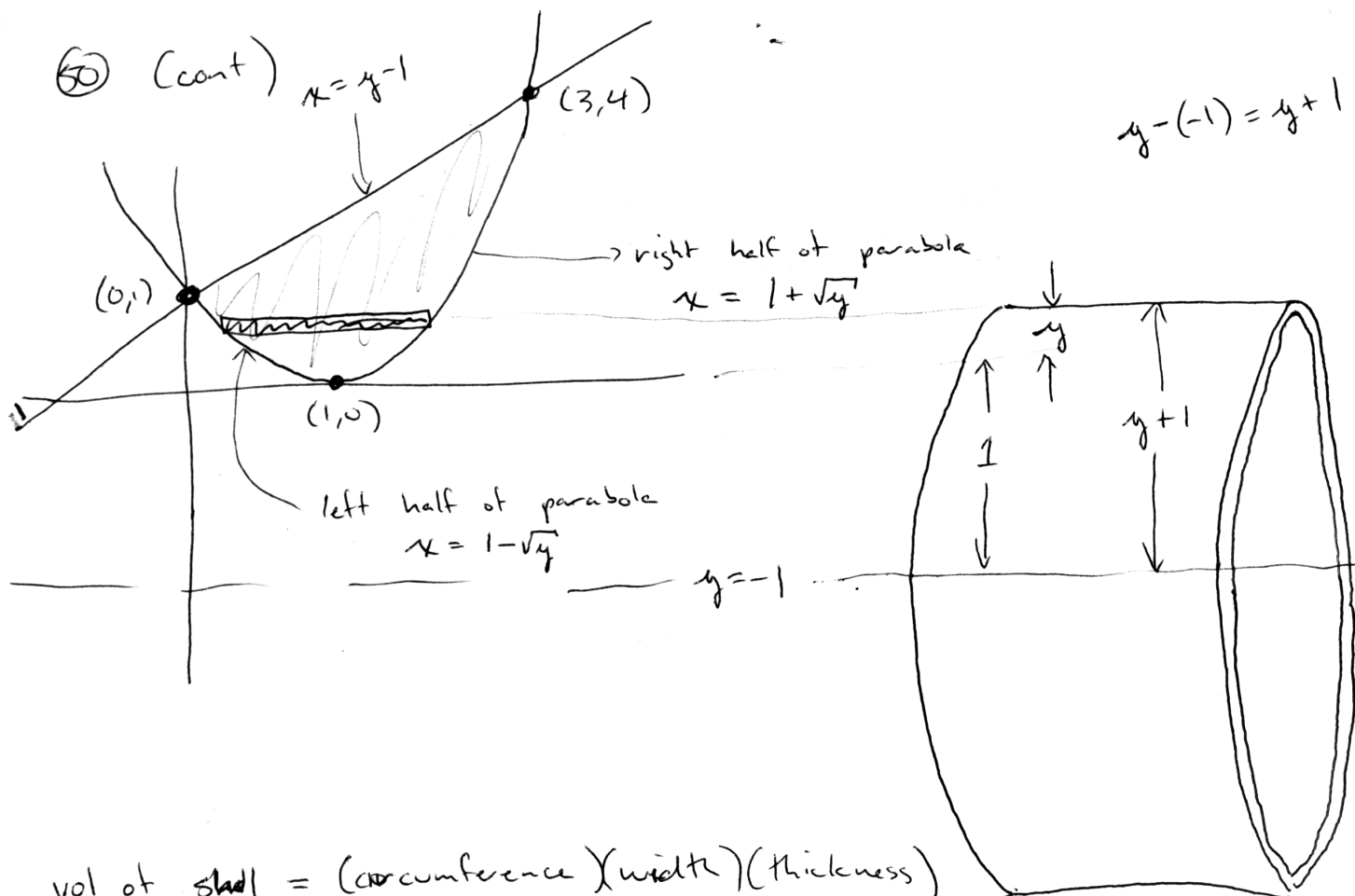
$$= \pi \left[(x+2)^2 - (x^2 - 2x + 2)^2 \right] dx$$

$$= \pi (-x^4 + 4x^3 - 7x^2 + 12x) dx$$

$$\text{total volume} = \int_0^3 \pi (-x^4 + 4x^3 - 7x^2 + 12x) dx$$

$$= \pi \left[-\frac{x^5}{5} + x^4 - \frac{7x^3}{3} + 6x^2 \right]_0^3$$

$$= \pi \left(-\frac{243}{5} + 81 - 63 + 54 \right) = \frac{127\pi}{5}$$



$$= \begin{cases} (2\pi(y+1)((1+\sqrt{y}) - (y-1)))(dy) & \text{if } 1 \leq y \leq 4 \\ (2\pi(y+1)((1+\sqrt{y}) - (1-\sqrt{y}))(dy) & \text{if } 0 \leq y \leq 1 \end{cases}$$

$$\text{total volume} = \int_0^1 (2\pi(y+1)(2\sqrt{y})) dy + \int_1^4 (2\pi(y+1)(-y + \sqrt{y} + 2)) dy$$

$$\int_0^1 2\pi(y+1)(2y^{1/2}) dy = 4\pi \int_0^1 (y^{3/2} + y^{1/2}) dy = 4\pi \left[\frac{2}{5} y^{5/2} + \frac{2}{3} y^{3/2} \right]_0^1 = 4\pi \left(\frac{2}{5} + \frac{2}{3} \right)$$

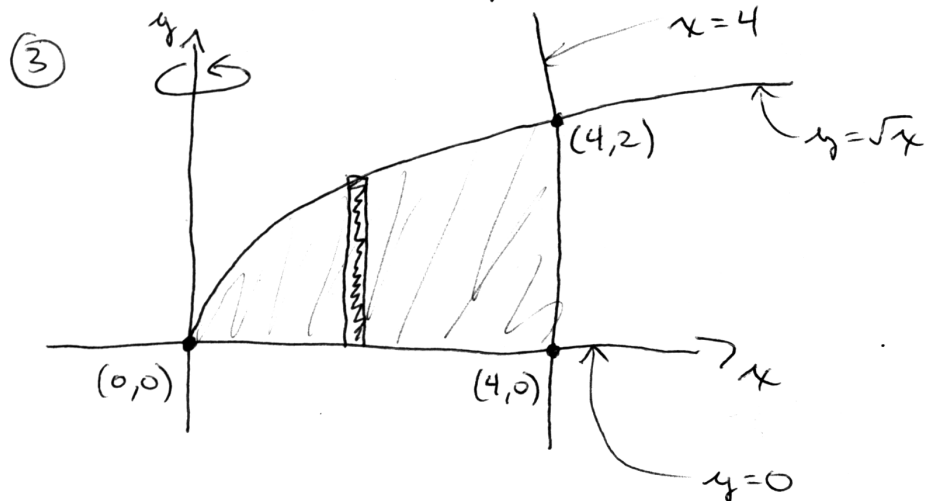
$$\begin{aligned} \int_1^4 2\pi(y+1)(-y + y^{1/2} + 2) dy &= 2\pi \int_1^4 (-y^2 + y^{3/2} + y + y^{1/2} + 2) dy \\ &= 2\pi \left[-\frac{1}{3} y^3 + \frac{2}{5} y^{5/2} + \frac{1}{2} y^2 + \frac{2}{3} y^{3/2} + 2y \right]_1^4 \\ &= 2\pi \left(-\frac{64}{3} + \frac{64}{5} + 8 + \frac{16}{3} + 8 \right) - 2\pi \left(-\frac{1}{3} + \frac{2}{5} + \frac{1}{2} + \frac{2}{3} + 2 \right) \\ &= 2\pi \left(\frac{52}{5} - \frac{1}{2} - \frac{1}{3} \right) \end{aligned}$$

$$\text{total volume} = 2\pi \left(\frac{52}{5} - \frac{1}{2} - \frac{1}{3} \right) + 4\pi \left(\frac{2}{5} + \frac{2}{3} \right) = \pi \left(\frac{104}{5} - 1 - \frac{2}{3} + \frac{8}{5} + \frac{8}{3} \right) = \frac{117\pi}{5}$$

6.3

3 & 5

Sketch the region bounded by the given curves, and find the volume of the solid swept out by ~~rotation~~ revolving the region about the y -axis.



vol of shell = $(2\pi \cdot \text{radius of shell})(\text{height of shell})(\text{thickness of shell})$

$$= (2\pi x)(\sqrt{x} - 0)(dx)$$

$$\text{total volume} = \int_0^4 (2\pi x)(x^{1/2}) dx$$

$$= 2\pi \left(\frac{2}{5} x^{5/2} \right) \Big|_0^4 = 2\pi \cdot \frac{2}{5} \cdot 32 - 0 = \frac{128\pi}{5}$$

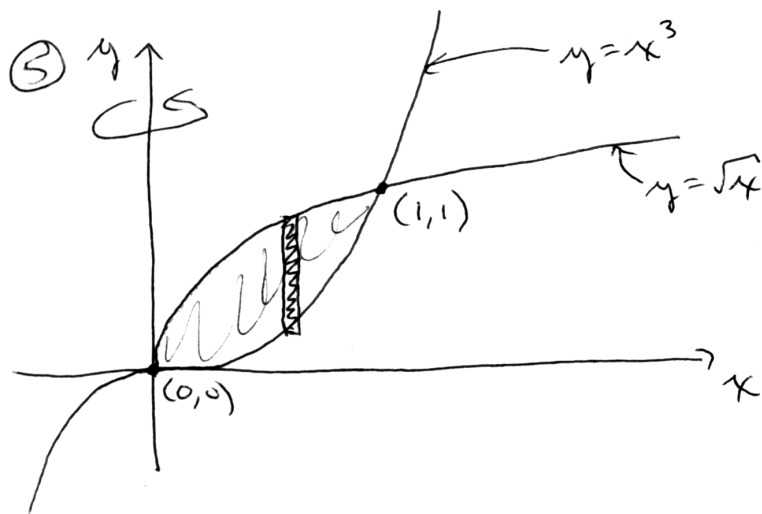
vol of washer = (area of top face)(thickness)

$$= (\pi \cdot 4^2 - \pi (y^2)^2)(dy)$$

$$\text{total volume} = \int_0^2 (16\pi - \pi y^4) dy$$

$$= 16\pi y - \frac{\pi}{5} y^5 \Big|_0^2 = 32\pi - \frac{32\pi}{5} - 0$$

$$= \frac{128\pi}{5}$$

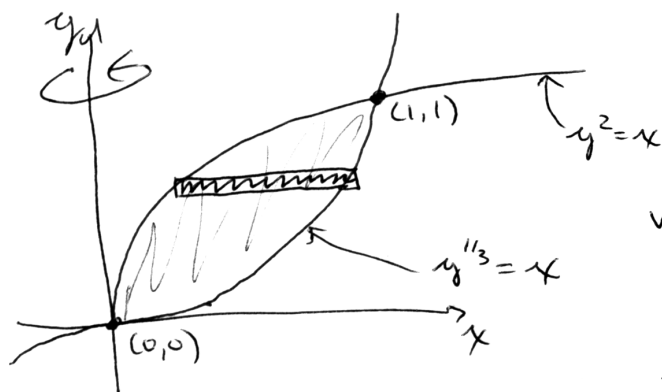


vol of washer = (circumference of shell)(height of shell)(thickness)

$$= (2\pi x)(\sqrt{x} - x^3)(dx)$$

$$\text{total volume} = \int_0^1 (2\pi x)(x^{1/2} - x^3) dx = 2\pi \int_0^1 (x^{3/2} - x^4) dx$$

$$= 2\pi \left[\frac{2}{5} x^{5/2} - \frac{1}{5} x^5 \right]_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{5} \right) - 0 = \frac{2\pi}{5}$$



volume of washer =

$$= (\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2)(\text{thickness})$$

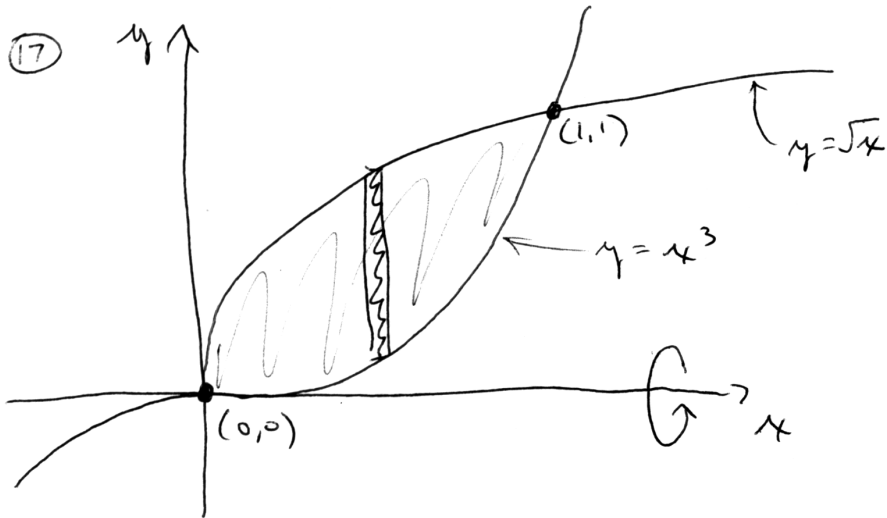
$$= (\pi(y^{5/3})^2 - \pi(y^2)^2)(dy)$$

$$\text{total volume} = \int_0^1 (\pi y^{2/3} - \pi y^4) dy$$

$$= \pi \left(\frac{3}{5} y^{5/3} - \frac{1}{5} y^5 \right) \Big|_0^1 = \pi \left(\frac{3}{5} - \frac{1}{5} \right) - 0 = \frac{2\pi}{5}$$

17 & 18

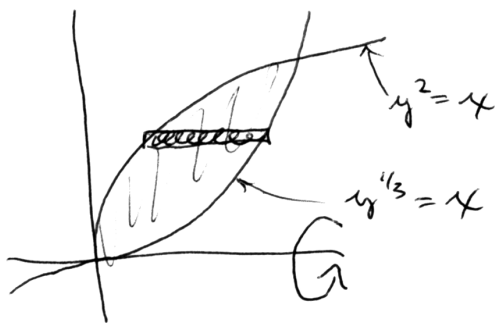
Sketch the region bounded by the given curves, and find the volume of the solid swept out by revolving the region around the x -axis.



$$\text{vol of washer} = (\pi(\sqrt{x})^2 - \pi(x^3)^2)(dx)$$

$$\text{total volume} = \int_0^1 (\pi x - \pi x^6) dx$$

$$= \left[\frac{\pi}{2} x^2 - \frac{\pi}{7} x^7 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{7} \right) - 0 = \frac{5\pi}{14}$$



$$\text{vol of shell} = (2\pi y)(y^{1/3} - y^2)(dy)$$

$$\text{total volume} = \int_0^1 (2\pi y)(y^{1/3} - y^2) dy$$

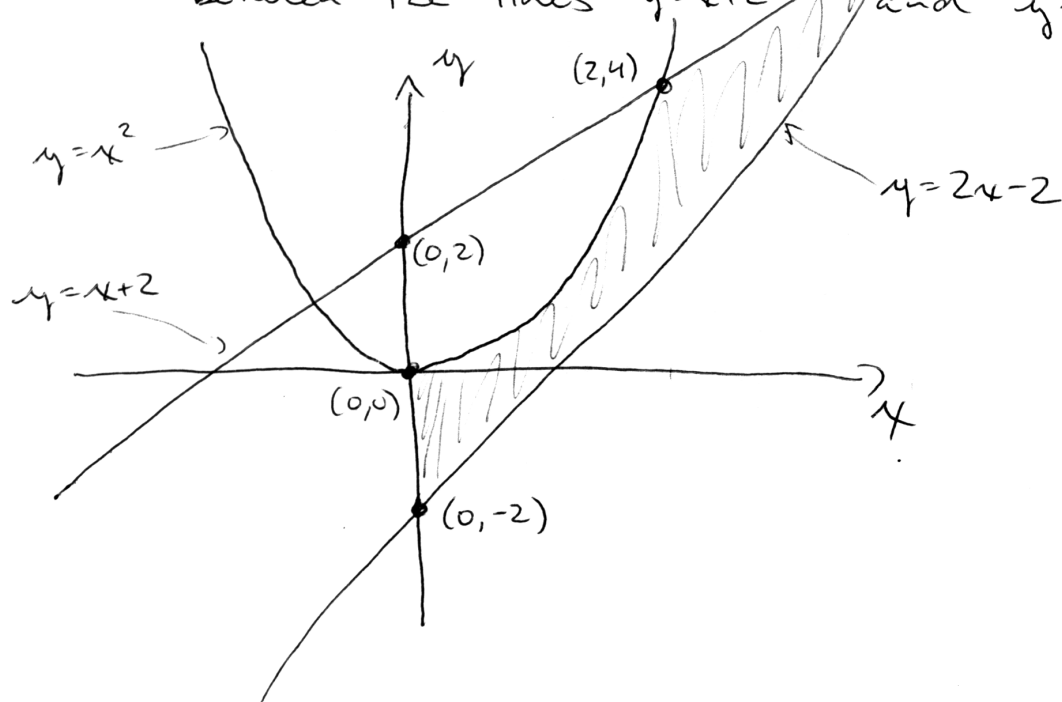
$$= 2\pi \int_0^1 (y^{4/3} - y^3) dy$$

$$= 2\pi \left[\frac{3}{7} y^{7/3} - \frac{1}{4} y^4 \right]_0^1$$

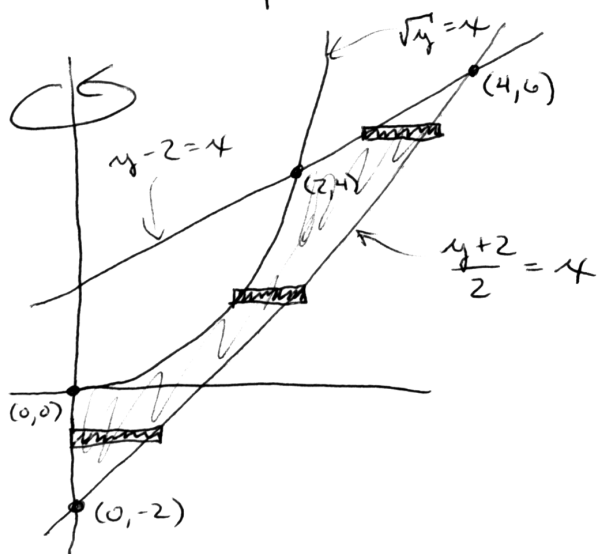
$$= 2\pi \left(\frac{3}{7} - \frac{1}{4} \right) - 0 = \frac{5\pi}{14}$$

①8 Same as ①7 but switch every x for every y & vice versa.

- 38) a) Sketch the region in the right half-plane that is outside the parabola $y = x^2$ and between the lines $y = x + 2$ and $y = 2x - 2$.



- b) The region above is revolved around the y -axis. Compute the volume swept out by the revolved region.



vol of washer =

$$(\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2)(\text{thickness})$$

↳ that is, ~~outer~~ further from the axis of rotation

↳ that is, closer to the axis of rotation

Note: This changes from one curve to another over the span of the solid

$$\text{vol of washer} = \begin{cases} (\pi(\frac{y+2}{2})^2 - \pi(\sqrt{y})^2)(dy) & \text{for } 0 \leq y \leq 4 \\ (\pi(\frac{y+2}{2})^2 - \pi(y-2)^2)(dy) & \text{for } 4 \leq y \leq 6 \\ (\pi(\frac{y+2}{2})^2 - \pi \cdot 0^2)(dy) & \text{for } -2 \leq y \leq 0 \end{cases}$$

38) (b) (cont)

$$\begin{aligned}\text{total volume} &= \int_{-2}^0 (\pi(\frac{y+2}{2})^2 - \pi 0^2) dy \\ &+ \int_0^4 (\pi(\frac{y+2}{2})^2 - \pi(\sqrt{y})^2) dy \\ &+ \int_4^6 (\pi(\frac{y+2}{2})^2 - \pi(y-2)^2) dy\end{aligned}$$

$$= \frac{\pi}{4} \int_{-2}^0 (y^2 + 4y + 4) dy$$

$$+ \pi \int_0^4 (\frac{1}{4}y^2 + 1) dy$$

$$+ \pi \int_4^6 (-\frac{3}{4}y^2 + 5y - 3) dy$$

$$= \frac{\pi}{4} \left[\frac{1}{3}y^3 + 2y^2 + 4y \right]_{-2}^0$$

$$+ \pi \left[\frac{1}{12}y^3 + y \right]_0^4$$

$$+ \pi \left[-\frac{1}{4}y^3 + \frac{5}{2}y^2 - 3y \right]_4^6$$

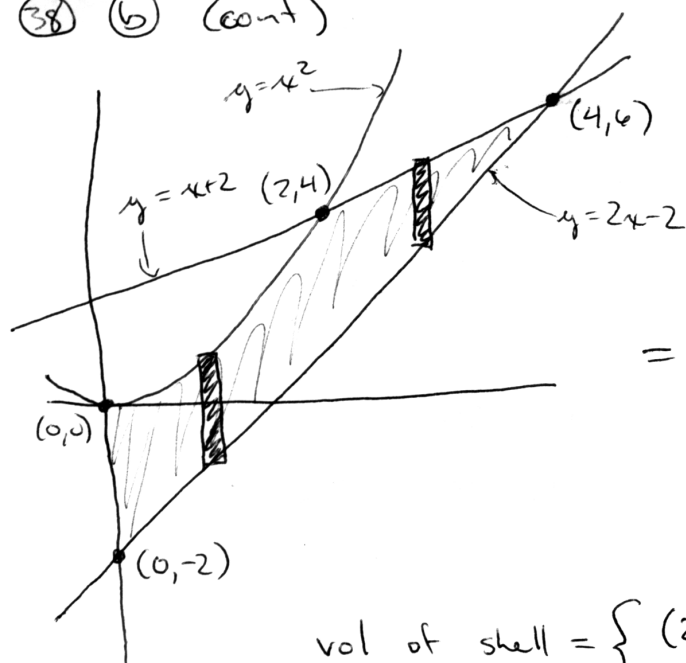
$$= \frac{\pi}{4} \left[(0) - \left(-\frac{8}{3} + 8 - 8 \right) \right]$$

$$+ \pi \left[\left(\frac{16}{3} + 4 \right) - (0) \right]$$

$$+ \pi \left[(-54 + 90 - 18) - (-16 + 40 - 12) \right]$$

$$= \frac{2}{3}\pi + \frac{28}{3}\pi + 6\pi = 16\pi$$

38) (b) (cont)



vol of a shell =

$$= (\text{circumference})(\text{height})(\text{thickness})$$

$$= (2\pi \cdot \text{radius of shell})(\text{top} - \text{bottom})(\text{thickness})$$

↑
this changes over the span of the region

$$\text{vol of shell} = \begin{cases} (2\pi x)((x^2) - (2x - 2))(dx) & \text{for } 0 \leq x \leq 2 \\ (2\pi x)((x + 2) - (2x - 2))(dx) & \text{for } 2 \leq x \leq 4 \end{cases}$$

$$\begin{aligned} \text{total volume} &= \int_0^2 (2\pi x)((x^2) - (2x - 2)) dx \\ &\quad + \int_2^4 (2\pi x)((x + 2) - (2x - 2)) dx \\ &= 2\pi \int_0^2 (x^3 - 2x^2 + 2x) dx \\ &\quad + 2\pi \int_2^4 (-x^2 + 4x) dx \\ &= 2\pi \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + x^2 \right]_0^2 \\ &\quad + 2\pi \left[-\frac{1}{3}x^3 + 2x^2 \right]_2^4 \\ &= 2\pi \left[\left(4 - \frac{16}{3} + 4 \right) - (0) \right] + 2\pi \left[\left(-\frac{64}{3} + 32 \right) - \left(-\frac{8}{3} + 8 \right) \right] \\ &= \frac{16}{3}\pi + \frac{32}{3}\pi = 16\pi \end{aligned}$$

I don't know about you, but I think the above method (the shell method) is easier for this problem.