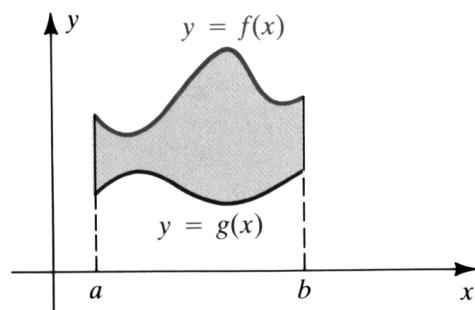


5.1

AREA



Figure 5.1



If a function f is continuous and $f(x) \geq 0$ on $[a, b]$, then, by Theorem (4.19), the area of the region under the graph of f from a to b is given by the definite integral $\int_a^b f(x) dx$. In this section, we consider the region that lies *between* the graphs of two functions.

If f and g are continuous and $f(x) \geq g(x) \geq 0$ for every x in $[a, b]$, then the area A of the region R bounded by the graphs of f , g , $x = a$, and $x = b$ (see Figure 5.1) can be found by subtracting the area of the region under the graph of g (the **lower boundary** of R) from the area of the region under the graph of f (the **upper boundary** of R), as follows:

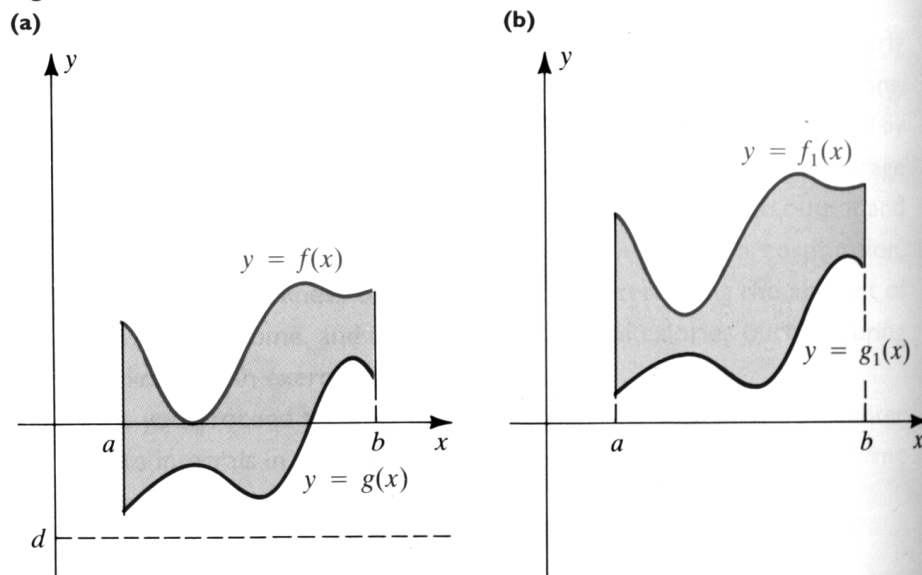
$$\begin{aligned} A &= \int_a^b f(x) dx - \int_a^b g(x) dx \\ &= \int_a^b [f(x) - g(x)] dx. \end{aligned}$$

This formula for A is also true if f or g is negative for some x in $[a, b]$. To verify this fact, choose a *negative* number d that is less than the minimum value of g on $[a, b]$, as illustrated in Figure 5.2(a). Next, consider the functions f_1 and g_1 , defined as follows:

$$\begin{aligned} f_1(x) &= f(x) - d = f(x) + |d| \\ g_1(x) &= g(x) - d = g(x) + |d| \end{aligned}$$

The graphs of f_1 and g_1 can be obtained by vertically shifting the graphs of f and g a distance $|d|$. If A is the area of the region in Figure 5.2(b),

Figure 5.2



then

$$\begin{aligned} A &= \int_a^b [f_1(x) - g_1(x)] dx \\ &= \int_a^b [(f(x) - d) - (g(x) - d)] dx \\ &= \int_a^b [f(x) - g(x)] dx. \end{aligned}$$

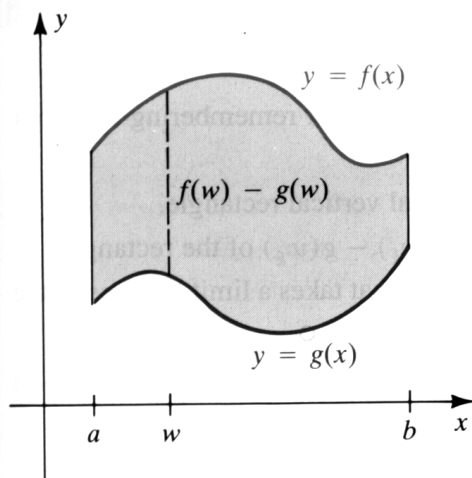
We may summarize our discussion as follows.

Theorem 5.1

If f and g are continuous and $f(x) \geq g(x)$ for every x in $[a, b]$, then the area A of the region bounded by the graphs of f , g , $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx.$$

Figure 5.3

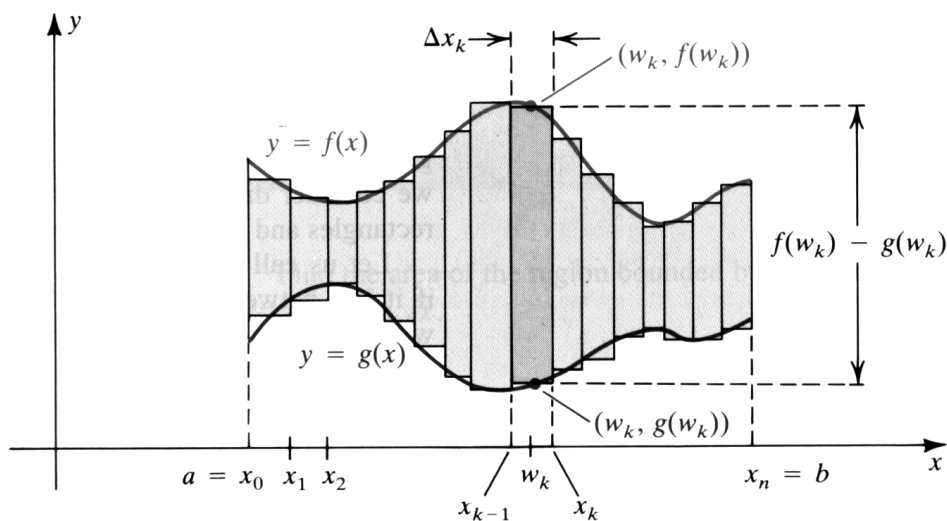


We may interpret the formula for A in Theorem (5.1) as a limit of sums. If we let $h(x) = f(x) - g(x)$ and if w is in $[a, b]$, then $h(w)$ is the vertical distance between the graphs of f and g for $x = w$ (see Figure 5.3). As in our discussion of Riemann sums in Chapter 4, let P denote a partition of $[a, b]$ determined by $a = x_0, x_1, \dots, x_n = b$. For each k , let $\Delta x_k = x_k - x_{k-1}$, and let w_k be any number in the k th subinterval $[x_{k-1}, x_k]$ of P . By the definition of h ,

$$h(w_k)\Delta x_k = [f(w_k) - g(w_k)]\Delta x_k,$$

which is the area of the rectangle of length $f(w_k) - g(w_k)$ and width Δx_k shown in Figure 5.4.

Figure 5.4



The Riemann sum

$$\sum_k h(w_k) \Delta x_k = \sum_k [f(w_k) - g(w_k)] \Delta x_k$$

is the sum of the areas of the rectangles in Figure 5.4 and is therefore an approximation to the area of the region between the graphs of f and g from a to b . By the definition of the definite integral,

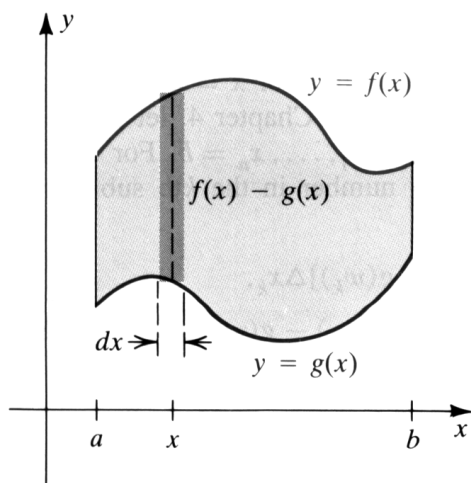
$$\lim_{\|P\| \rightarrow 0} \sum_k h(w_k) \Delta x_k = \int_a^b h(x) dx.$$

Since $h(x) = f(x) - g(x)$, we obtain the following corollary of Theorem (5.1).

Corollary 5.2

$$A = \lim_{\|P\| \rightarrow 0} \sum_k [f(w_k) - g(w_k)] \Delta x_k = \int_a^b [f(x) - g(x)] dx$$

Figure 5.5



We may use the following intuitive method for remembering this limit of sums formula (see Figure 5.5):

1. Use dx for the width Δx_k of a typical vertical rectangle.
2. Use $f(x) - g(x)$ for the length $f(w_k) - g(w_k)$ of the rectangle.
3. Regard the symbol \int_a^b as an operator that takes a limit of sums of the rectangular areas $[f(x) - g(x)] dx$.

This method allows us to interpret the area formula in Theorem (5.1) as follows:

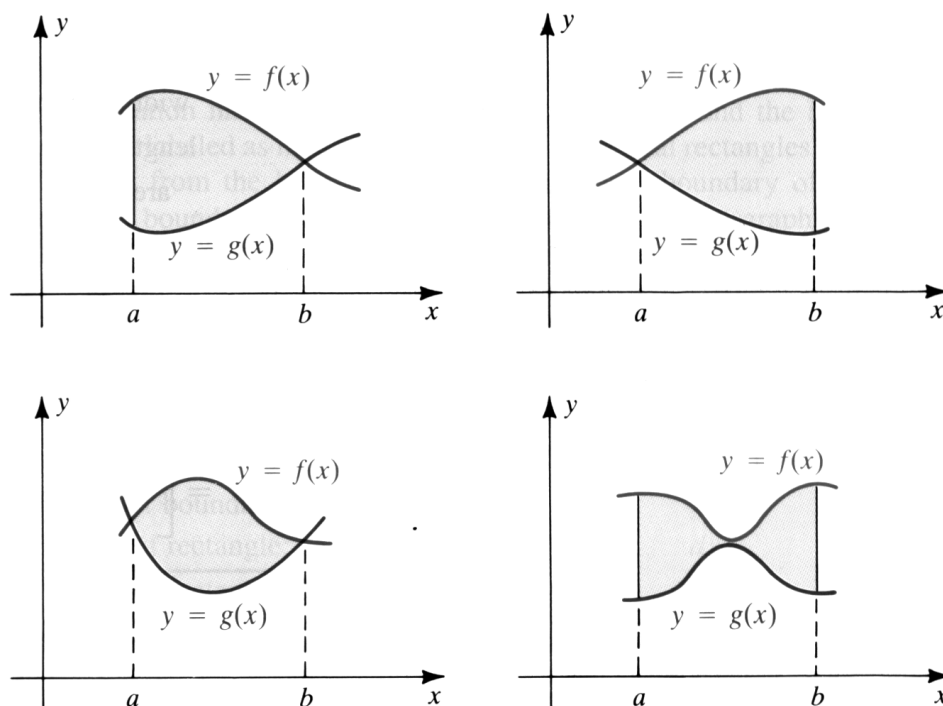
$$A = \int_a^b [f(x) - g(x)] dx$$

limit of sums
length of a rectangle
width of a rectangle

When using this technique, we visualize summing areas of vertical rectangles by moving through the region from left to right. Later in this section, we consider different types of regions, finding areas by using *horizontal* rectangles and integrating with respect to y .

Let us call a region an **R_x region** (for integration with respect to x) if it lies between the graphs of two equations $y = f(x)$ and $y = g(x)$, with f and g continuous, and $f(x) \geq g(x)$ for every x in $[a, b]$, where a and b are the smallest and largest x -coordinates, respectively, of the points (x, y) in the region. The regions in Figures 5.1–5.5 are R_x regions. Several others are sketched in Figure 5.6 on the following page. Note that the graphs of $y = f(x)$ and $y = g(x)$ may intersect one or more times; however, $f(x) \geq g(x)$ throughout the interval.

Figure 5.6 R_x regions

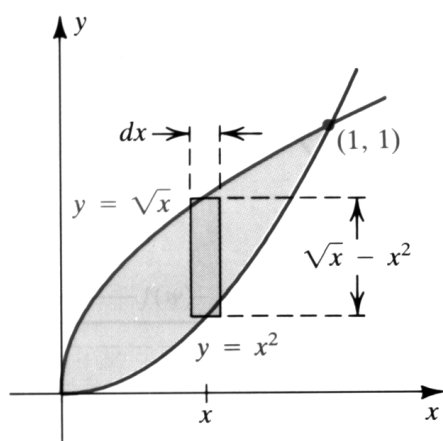


The following guidelines may be helpful when working problems.

Guidelines for Finding the Area of an R_x Region 5.3

- 1 Sketch the region, labeling the upper boundary $y = f(x)$ and the lower boundary $y = g(x)$. Find the smallest value $x = a$ and the largest value $x = b$ for points (x, y) in the region.
- 2 Sketch a typical vertical rectangle and label its width dx .
- 3 Express the area of the rectangle in guideline (2) as $[f(x) - g(x)] dx$.
- 4 Apply the limit of sums operator \int_a^b to the expression in guideline (3) and evaluate the integral.

Figure 5.7



EXAMPLE ■ I Find the area of the region bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$.

SOLUTION Following guidelines (1)–(3), we sketch and label the region and show a typical vertical rectangle (see Figure 5.7). The points $(0, 0)$ and $(1, 1)$ at which the graphs intersect can be found by solving the equations $y = x^2$ and $y = \sqrt{x}$ simultaneously. Referring to the figure, we

obtain the following facts:

$$\text{upper boundary: } y = \sqrt{x}$$

$$\text{lower boundary: } y = x^2$$

$$\text{width of rectangle: } dx$$

$$\text{length of rectangle: } \sqrt{x} - x^2$$

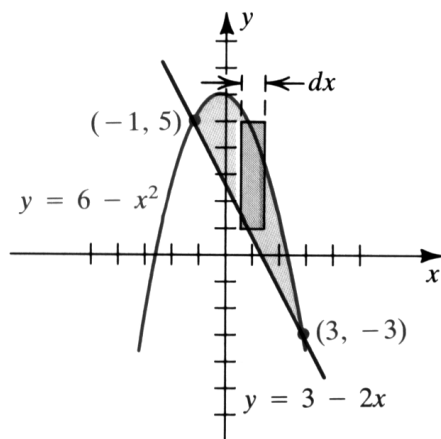
$$\text{area of rectangle: } (\sqrt{x} - x^2) dx$$

Next, we follow guideline (4) with $a = 0$ and $b = 1$, remembering that applying \int_0^1 to the expression $(\sqrt{x} - x^2) dx$ represents taking a limit of sums of areas of vertical rectangles. We thus obtain

$$\begin{aligned} A &= \int_0^1 (\sqrt{x} - x^2) dx = \int_0^1 (x^{1/2} - x^2) dx \\ &= \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}. \end{aligned}$$

EXAMPLE ■ 2 Find the area of the region bounded by the graphs of $y + x^2 = 6$ and $y + 2x - 3 = 0$.

Figure 5.8



SOLUTION The region and a typical rectangle are sketched in Figure 5.8. The points of intersection $(-1, 5)$ and $(3, -3)$ of the two graphs may be found by solving the two given equations simultaneously. To apply guideline (1), we must label the upper and lower boundaries $y = f(x)$ and $y = g(x)$, respectively, and hence we solve each of the given equations for y in terms of x , as shown in Figure 5.8. Here we obtain

$$\text{upper boundary: } y = 6 - x^2$$

$$\text{lower boundary: } y = 3 - 2x$$

$$\text{width of rectangle: } dx$$

$$\text{length of rectangle: } (6 - x^2) - (3 - 2x)$$

$$\text{area of rectangle: } [(6 - x^2) - (3 - 2x)] dx$$

Next, we use guideline (4), with $a = -1$ and $b = 3$, regarding \int_{-1}^3 as an operator that takes a limit of sums of areas of rectangles. Thus,

$$\begin{aligned} A &= \int_{-1}^3 [(6 - x^2) - (3 - 2x)] dx = \int_{-1}^3 (3 - x^2 + 2x) dx \\ &= \left[3x - \frac{x^3}{3} + x^2 \right]_{-1}^3 \\ &= [9 - \frac{27}{3} + 9] - [-3 - (-\frac{1}{3}) + 1] = \frac{32}{3}. \end{aligned}$$

The following example illustrates that it is sometimes necessary to subdivide a region into several R_x regions and then use more than one definite integral to find the area.

EXAMPLE ■ 3 Find the area of the region R bounded by the graphs of $y - x = 6$, $y - x^3 = 0$, and $2y + x = 0$.

Figure 5.9

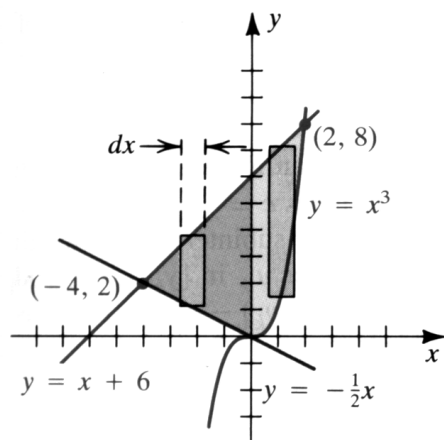


Figure 5.10

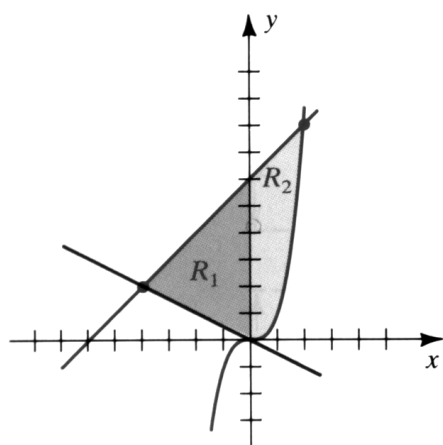
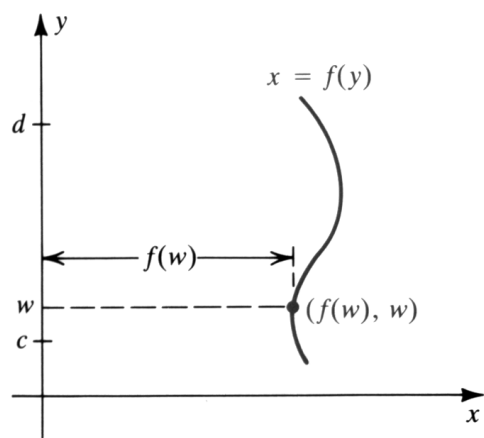


Figure 5.11



SOLUTION The graphs and the region are sketched in Figure 5.9. Each equation has been solved for y in terms of x , and the boundaries have been labeled as in guideline (1). Typical vertical rectangles are shown extending from the lower boundary to the upper boundary of R . Since the lower boundary consists of portions of two different graphs, the area cannot be found by using only one definite integral. However, if R is divided into two R_x regions, R_1 and R_2 , as shown in Figure 5.10, then we can determine the area of each and add them together. Let us arrange our work as follows.

	Region R_1	Region R_2
upper boundary:	$y = x + 6$	$y = x + 6$
lower boundary:	$y = -\frac{1}{2}x$	$y = x^3$
width of rectangle:	dx	dx
length of rectangle:	$(x + 6) - (-\frac{1}{2}x)$	$(x + 6) - x^3$
area of rectangle:	$[(x + 6) - (-\frac{1}{2}x)] dx$	$[(x + 6) - x^3] dx$

Applying guideline (4), we find the areas A_1 and A_2 of R_1 and R_2 :

$$\begin{aligned}
 A_1 &= \int_{-4}^0 [(x + 6) - (-\frac{1}{2}x)] dx \\
 &= \int_{-4}^0 \left(\frac{3}{2}x + 6 \right) dx = \left[\frac{3}{2} \left(\frac{x^2}{2} \right) + 6x \right]_{-4}^0 \\
 &= 0 - (12 - 24) = 12 \\
 A_2 &= \int_0^2 [(x + 6) - x^3] dx \\
 &= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4} \right]_0^2 \\
 &= (2 + 12 - 4) - 0 = 10
 \end{aligned}$$

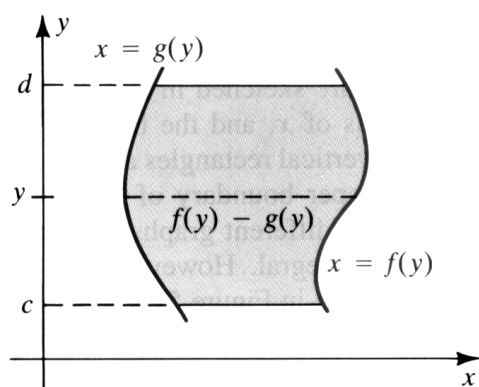
The area A of the entire region R is

$$A = A_1 + A_2 = 12 + 10 = 22.$$

We have now evaluated many integrals similar to those in Example 3. For this reason, we sometimes merely *set up* an integral—that is, we express it in the proper form but do not find its numerical value.

If we consider an equation of the form $x = f(y)$, where f is continuous for $c \leq y \leq d$, then we *reverse the roles of x and y in the previous discussion, treating y as the independent variable and x as the dependent variable*. A typical graph of $x = f(y)$ is sketched in Figure 5.11. Note that if a value w is assigned to y , then $f(w)$ is an x -coordinate of the corresponding point on the graph.

Figure 5.12



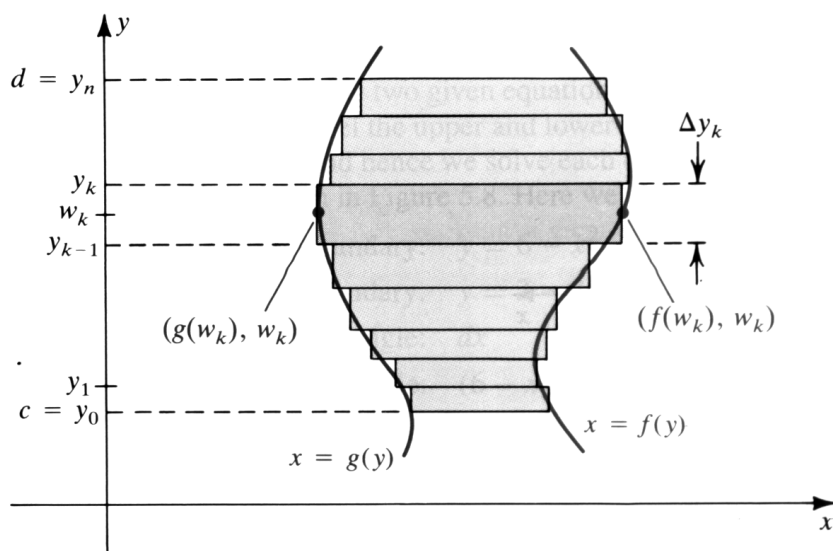
An R_y region is a region that lies between the graphs of two equations of the form $x = f(y)$ and $x = g(y)$, with f and g continuous, and with $f(y) \geq g(y)$ for every y in $[c, d]$, where c and d are the smallest and largest y -coordinates, respectively, of points in the region. One such region is illustrated in Figure 5.12. We call the graph of f the **right boundary** of the region and the graph of g the **left boundary**. For any y , the number $f(y) - g(y)$ is the horizontal distance between these boundaries, as shown in Figure 5.12.

We can use limits of sums to find the area A of an R_y region. We begin by selecting points on the y -axis with y -coordinates $c = y_0, y_1, \dots, y_n = d$, obtaining a partition of the interval $[c, d]$ into subintervals of width $\Delta y_k = y_k - y_{k-1}$. For each k , we choose a number w_k in $[y_{k-1}, y_k]$ and consider horizontal rectangles that have areas $[f(w_k) - g(w_k)]\Delta y_k$, as illustrated in Figure 5.13. This procedure leads to

$$A = \lim_{\|P\| \rightarrow 0} \sum_k [f(w_k) - g(w_k)]\Delta y_k = \int_c^d [f(y) - g(y)] dy.$$

The last equality follows from the definition of the definite integral.

Figure 5.13



Using notation similar to that for R_x regions, we represent the width Δy_k of a horizontal rectangle by dy and the length $f(w_k) - g(w_k)$ of the rectangle by $f(y) - g(y)$ in the following guidelines.

Guidelines for Finding the Area of an R_y Region 5.4

- 1 Sketch the region, labeling the right boundary $x = f(y)$ and the left boundary $x = g(y)$. Find the smallest value $y = c$ and the largest value $y = d$ for points (x, y) in the region.
- 2 Sketch a typical horizontal rectangle and label its width dy .

3 Express the area of the rectangle in guideline (2) as

$$[f(y) - g(y)] dy.$$

4 Apply the limit of sums operator \int_c^d to the expression in guideline (3) and evaluate the integral.

In guideline (4), we visualize summing areas of horizontal rectangles by moving from the lowest point of the region to the highest point.

Figure 5.14

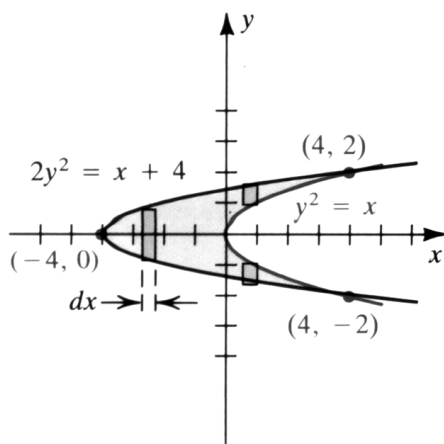
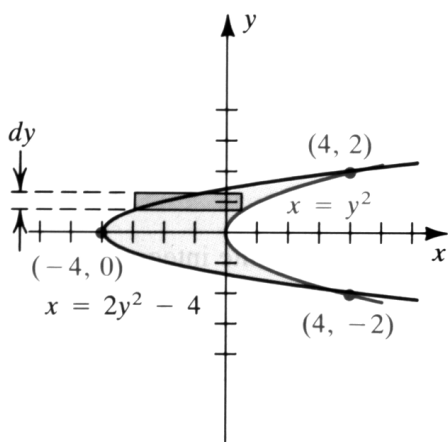


Figure 5.15



EXAMPLE ■ 4 Find the area of the region bounded by the graphs of the equations $2y^2 = x + 4$ and $y^2 = x$.

SOLUTION The region is sketched in Figures 5.14 and 5.15. Figure 5.14 illustrates the use of vertical rectangles (integration with respect to x), and Figure 5.15 illustrates the use of horizontal rectangles (integration with respect to y). Referring to Figure 5.14, we see that several integrations with respect to x are required to find the area. However, for Figure 5.15, we need only one integration with respect to y . Thus we apply Guidelines (5.4), solving each equation for x in terms of y . Referring to Figure 5.15, we obtain the following:

$$\text{right boundary: } x = y^2$$

$$\text{left boundary: } x = 2y^2 - 4$$

$$\text{width of rectangle: } dy$$

$$\text{length of rectangle: } y^2 - (2y^2 - 4)$$

$$\text{area of rectangle: } [y^2 - (2y^2 - 4)] dy$$

We could now use guideline (4) with $c = -2$ and $d = 2$, finding A by applying the operator \int_{-2}^2 to $[y^2 - (2y^2 - 4)] dy$. Another method is to use the symmetry of the region with respect to the x -axis and find A by doubling the area of the part that lies above the x -axis. Thus,

$$\begin{aligned} A &= \int_{-2}^2 [y^2 - (2y^2 - 4)] dy \\ &= 2 \int_0^2 (4 - y^2) dy \\ &= 2 \left[4y - \frac{y^3}{3} \right]_0^2 = 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}. \end{aligned}$$

In following Guidelines (5.3) or (5.4) for finding the area of a region, we may need to use a graphing utility and numerical methods to obtain an accurate sketch of the region, find the smallest and largest x - or y -values in the region, and approximate the area. Our next example illustrates such a case.



EXAMPLE ■ 5 For the region of the plane bounded by the curves $y = \cos(0.3x^2)$ and $y = x^2 + 0.6x - 2$,

- (a) use a graphing utility to sketch the curves and determine the region
- (b) find numerical approximations for the intersection points of the bounding curves
- (c) set up a definite integral representing the area of the region
- (d) approximate the area

Figure 5.16

$$-3.8 \leq x \leq 3.8, \quad -2.5 \leq y \leq 2$$

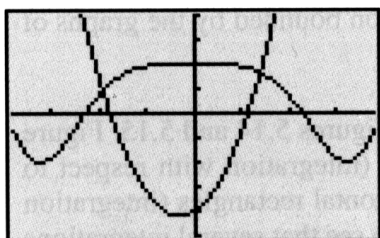
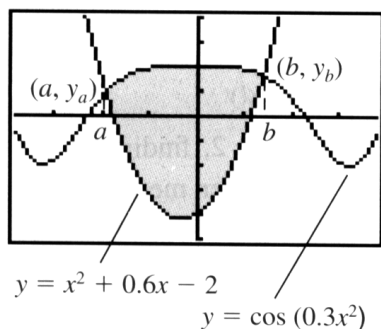


Figure 5.17



SOLUTION

(a) After examining several different viewing windows, we obtain the view of the desired area shown in Figure 5.16.

(b) By tracing on the graph, we obtain first approximations for the intersection points as $(-1.9, 0.5)$ and $(1.4, 0.8)$. At the intersection points, the bounding curves have equal y -values. Thus, $\cos(0.3x^2) = x^2 + 0.6x - 2$, so $\cos(0.3x^2) - [x^2 + 0.6x - 2] = 0$. We can use a solving routine or apply Newton's method to the function

$$f(x) = \cos(0.3x^2) - x^2 - 0.6x + 2$$

with starting values of $x = -1.9$ and then $x = 1.4$ to obtain values $a \approx -1.89968629228$ and $b \approx 1.40826496779$. Substituting these values into the equation for either bounding curve gives the other coordinates for the points of intersection, $y_a \approx 0.468996233702$ and $y_b \approx 0.828169200187$.

(c) In Figure 5.17, we have labeled the bounding curves and points of intersection and shaded the region. We see there that the graph of $y = \cos(0.3x^2)$ is above the graph of $y = x^2 + 0.6x - 2$ on the interval $[a, b]$. The integral representing the area of the region is

$$\begin{aligned} A &= \int_a^b [\cos(0.3x^2) - (x^2 + 0.6x - 2)] dx \\ &\approx \int_{-1.89968629228}^{1.40826496779} [\cos(0.3x^2) - x^2 - 0.6x + 2] dx. \end{aligned}$$

(d) We compute numerical approximations for the definite integral in part (c) using Simpson's rule for several different values of n to approximate the area. For example, when $n = 64$, $A \approx 6.93542681443$ and when $n = 128$, $A \approx 6.93542681577$. Thus, we have confidence in the approximation that, to seven decimal places, $A \approx 6.9354268$.

As an application of the area between two curves, let us consider what economists call **capital formation**—that is, the process of increasing or decreasing a given holding of capital over time. If $K(t)$ is the amount of capital at time t , then dK/dt denotes the **rate of capital formation**. Economists consider the rate of capital formation to be identical to the **net investment flow**, which we will denote by $I(t)$. We can look at the

relationship between capital formation and net investment flow in two ways: in a derivative formulation,

$$\frac{dK}{dt} = I(t)$$

and in an integral form,

$$K(t) = \int I(t) dt.$$

Note that with $I(t) \geq 0$ for $a \leq t \leq b$, the amount of capital accumulation in this time interval is $\int_a^b I(t) dt$, the area under the graph of the function $I(t)$.

If we know the amount of capital $K(t)$ accumulated at time t , we may differentiate with respect to t to find the investment flow. Alternatively, if we are given the investment flow $I(t)$, we may integrate with respect to t to find the amount of capital—that is, $K(t)$ represents the total change in capital or the capital accumulation. As a derivative, the investment flow $I(t)$ is a rate of change of capital. That is, the value of $I(t)$ at a particular time t is the rate at which investment is flowing in or out of the given holding of capital, measured in units of capital per unit of time. For example, if $I(t) = 4 - t^2 + 2t$, where capital is measured in millions of dollars and time is measured in years, at time $t = 1$ year, we have $I(1) = 4 - 1 + 2 = 5$ million dollars per year. Hence, capital is increasing at an annual rate of \$5 million. At $t = 4$, $I(4) = 4 - 16 + 8 = -4$, so at time $t = 4$ years, capital is decreasing at an annual rate of \$4 million.

EXAMPLE ■ 6 If the net investment flow is $I(t) = 4 - t^2 + 2t$ millions of dollars per time unit, find the capital formation during the time interval $[1, 2]$.

SOLUTION The capital formation is given by

$$\int_1^2 I(t) dt = \int_1^2 (4 - t^2 + 2t) dt.$$

We can evaluate the definite integral by finding an antiderivative for $I(t)$:

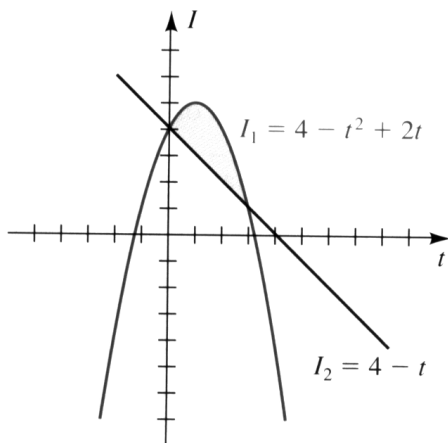
$$\begin{aligned} \int_1^2 (4 - t^2 + 2t) dt &= \left[4t - \frac{t^3}{3} + t^2 \right]_1^2 \\ &= \left[8 - \frac{8}{3} + 4 \right] - \left[4 - \frac{1}{3} + 1 \right] = 4\frac{2}{3} \end{aligned}$$

Thus, the capital accumulation is about \$4.67 million.

EXAMPLE ■ 7 Consider two different net investment flows given by $I_1(t) = 4 - t^2 + 2t$ and $I_2(t) = 4 - t$ (both in millions of dollars per year at year t).

(a) Find the time interval during which the first investment flow I_1 is at least as great as the second investment flow I_2 .

Figure 5.18



(b) For the time interval found in part (a), determine how much more capital accumulates under the first investment flow than under the second investment flow.

SOLUTION

(a) We need to find the interval $[a, b]$ during which $I_1(t) \geq I_2(t)$. We first sketch the graphs of the two functions (Figure 5.18) and then find the points of intersection by solving the equations $y = 4 - t^2 + 2t$ and $y = 4 - t$ simultaneously:

$$\begin{aligned} 4 - t^2 + 2t &= 4 - t \\ 3t - t^2 &= 0 \\ t(3 - t) &= 0 \\ t = 0 \quad \text{and} \quad t = 3 \end{aligned}$$

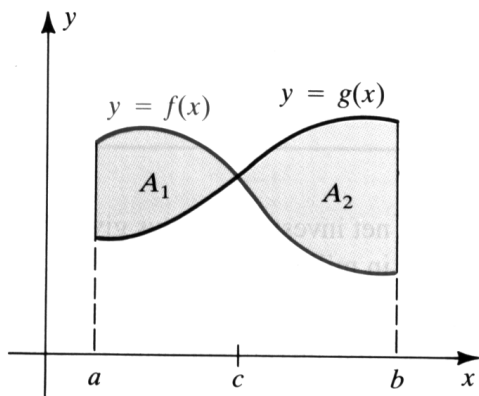
Thus, we see from the graph that $I_1(t) \geq I_2(t)$ on the interval $[0, 3]$. If $t = 0$ corresponds to the present time, then the investment flow I_1 will exceed the investment flow I_2 for the next three years.

(b) The difference in capital accumulation between the two investment flows is the area of the region between the two curves I_1 and I_2 over the interval $[0, 3]$. That is,

$$\begin{aligned} \int_0^3 [I_1(t) - I_2(t)] dt &= \int_0^3 [4 - t^2 + 2t - (4 - t)] dt \\ &= \int_0^3 [3t - t^2] dt \\ &= \left[\frac{3t^2}{2} - \frac{t^3}{3} \right]_0^3 = \left[\frac{27}{2} - \frac{27}{3} \right] - [0 - 0] = \frac{9}{2}. \end{aligned}$$

Thus, the first investment flow will generate \$4.5 million more in accumulated capital than the second investment flow during the next three years.

Figure 5.19



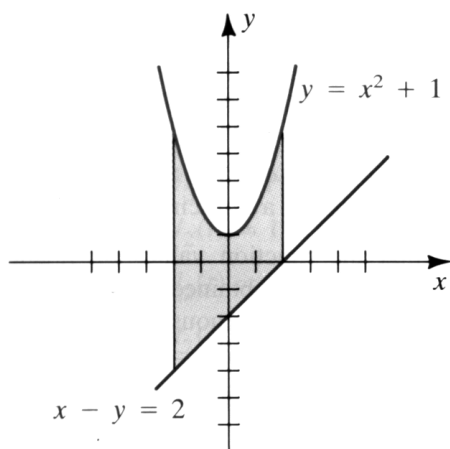
Throughout this section, we have assumed that the graphs of the functions (or equations) do not cross one another in the interval under discussion. If the graphs of f and g cross at one point $P(c, d)$, with $a < c < b$, and we wish to find the area bounded by the graphs from $x = a$ to $x = b$, then the methods developed in this section may still be used; however, *two* integrations are required, one corresponding to the interval $[a, c]$ and the other to $[c, b]$, as is illustrated in Figure 5.19, with $f(x) \geq g(x)$ on $[a, c]$ and $g(x) \geq f(x)$ on $[c, b]$. The area A is given by

$$A = A_1 + A_2 = \int_a^c [f(x) - g(x)] dx + \int_c^b [g(x) - f(x)] dx.$$

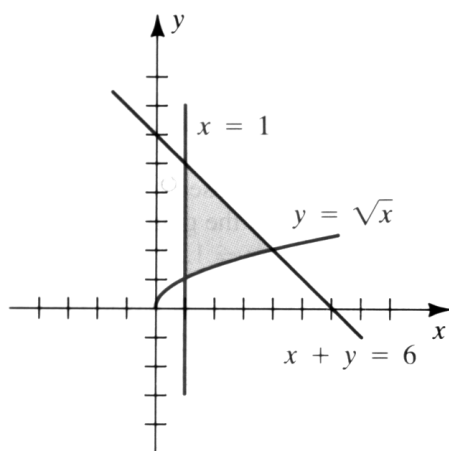
If the graphs cross *several* times, then several integrals may be necessary. Problems in which graphs cross one or more times appear in Exercises 31–36.

Exer. 1–4: Set up an integral that can be used to find the area of the shaded region.

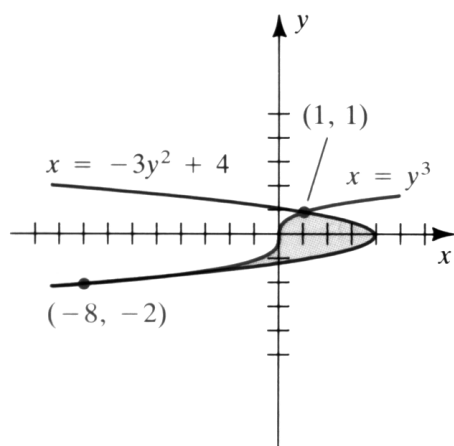
1



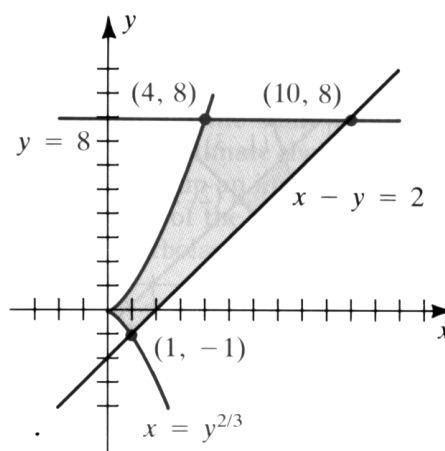
2



3



4



Exer. 5–22: Sketch the region bounded by the graphs of the equations and find its area.

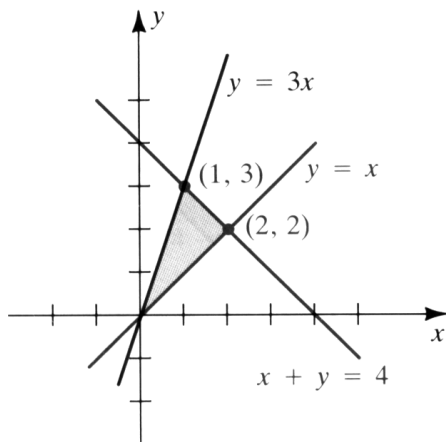
- 5 $y = x^2$; $y = 4x$
- 6 $x + y = 3$; $y + x^2 = 3$
- 7 $y = x^2 + 1$; $y = 5$
- 8 $y = 4 - x^2$; $y = -4$
- 9 $y = 1/x^2$; $y = -x^2$; $x = 1$; $x = 2$
- 10 $y = x^3$; $y = x^2$
- 11 $y^2 = -x$; $x - y = 4$; $y = -1$; $y = 2$
- 12 $x = y^2$; $y - x = 2$; $y = -2$; $y = 3$
- 13 $y^2 = 4 + x$; $y^2 + x = 2$
- 14 $x = y^2$; $x - y = 2$
- 15 $x = 4y - y^3$; $x = 0$
- 16 $x = y^{2/3}$; $x = y^2$
- 17 $y = x^3 - x$; $y = 0$
- 18 $y = x^3 - x^2 - 6x$; $y = 0$
- 19 $x = y^3 + 2y^2 - 3y$; $x = 0$
- 20 $x = 9y - y^3$; $x = 0$
- 21 $y = x\sqrt{4 - x^2}$; $y = 0$
- 22 $y = x\sqrt{x^2 - 9}$; $y = 0$; $x = 5$

Exer. 23–24: Find the area of the region between the graphs of the two equations from $x = 0$ to $x = \pi$.

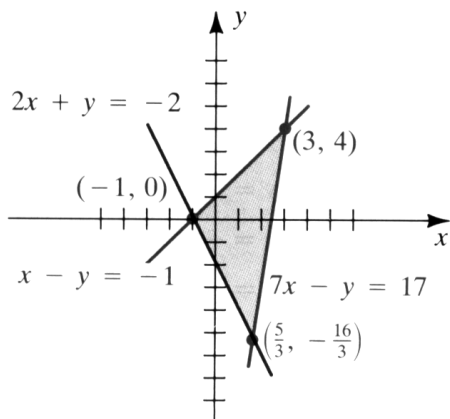
- 23 $y = \sin 4x$; $y = 1 + \cos \frac{1}{3}x$
- 24 $y = 4 + \cos 2x$; $y = 3 \sin \frac{1}{2}x$

Exer. 25–26: Set up sums of integrals that can be used to find the area of the shaded region by integrating with respect to (a) x and (b) y .

25



26



Exer. 27–30: Set up sums of integrals that can be used to find the area of the region bounded by the graphs of the equations by integrating with respect to (a) x and (b) y .

27 $y = \sqrt{x}$; $y = -x$; $x = 1$; $x = 4$

28 $y = 1 - x^2$; $y = x - 1$

29 $y = x + 3$; $x = -y^2 + 3$

30 $x = y^2$; $x = 2y^2 - 4$

Exer. 31–36: Find the area of the region between the graphs of f and g if x is restricted to the given interval.

31 $f(x) = 6 - 3x^2$; $g(x) = 3x$; $[0, 2]$

32 $f(x) = x^2 - 4$; $g(x) = x + 2$; $[1, 4]$

33 $f(x) = x^3 - 4x + 2$; $g(x) = 2$; $[-1, 3]$

34 $f(x) = x^2$; $g(x) = x^3$; $[-1, 2]$

35 $f(x) = \sin x$; $g(x) = \cos x$; $[0, 2\pi]$

36 $f(x) = \sin x$; $g(x) = \frac{1}{2}$; $[0, \pi/2]$

Exer. 37–38: Let R be the region bounded by the graph of f and the x -axis, from $x = a$ to $x = b$. Set up a sum of integrals, not containing the absolute value symbol, that can be used to find the area of R .

37 $f(x) = |x^2 - 6x + 5|$; $a = 0$, $b = 7$

38 $f(x) = |-x^2 + 2x + 3|$; $a = -3$, $b = 4$

39 Show that the area of the region bounded by an ellipse whose major and minor axes have lengths $2a$ and $2b$, respectively, is πab . (Hint: Use an equation of the ellipse to show first that the area is given by $2(b/a) \int_{-a}^a \sqrt{a^2 - x^2} dx$, and then interpret the definite integral as the area of a semicircle of radius a .)

40 Suppose that the function values of f and g in the following table were obtained empirically. Assuming that f and g are continuous, approximate the area between their graphs from $x = 1$ to $x = 5$ using (a) the trapezoidal rule, with $n = 8$, and (b) Simpson's rule, with $n = 4$.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	3.5	2.5	3	4	3.5	2.5	2	2	3
$g(x)$	1.5	2	2	1.5	1	0.5	1	1.5	1

c 41 Graph $f(x) = |x^3 - 0.7x^2 - 0.8x + 1.3|$ on $[-1.5, 1.5]$. Set up a sum of integrals, not containing the absolute value symbol, that can be used to approximate the area of the region bounded by the graph of f , the x -axis, and the lines $x = -1.5$ and $x = 1.5$.

c 42 Graph, on the same coordinate axes, $f(x) = \sin x$ and $g(x) = x^3 - x + 0.2$ for $-2 \leq x \leq 2$. Set up a sum of integrals that can be used to approximate the area of the region bounded by the graphs.

c Exer. 43–46: Plot the graphs of the equations. (a) Find numerical approximations for the intersection points of the different bounding curves. (b) Set up a definite integral representing the area of the bounded region. (c) Approximate this area to four-decimal-place accuracy using Simpson's rule.

43 $y = x^3 - 2x^2 - x + 1$; $y = \sqrt{10x}$

44 $y = 4x^4 - 8x^2 + x - 1$; $y = -2x^2 - x + 4$

45 $y = 50 \cos(0.5x)$; $y = x^2 - 20$

46 $y = 0.2x^4 - x^3 + 0.4x^2 - 2$; $y = \cos(0.7x)$

c Exer. 47–50: Plot the graphs of the equations. (a) Set up a definite integral representing the area of the bounded region. (b) Approximate this area to four-decimal-place accuracy using Simpson's rule.

47 $y = \sqrt{25 - x^2}$; $y = \sqrt{29 - x^2} - 2$

$$48 \quad y = \sin[\pi(x^2 - 1)]; \quad y = 1 - x^2$$

$$49 \quad y = \sin x; \quad y = \sin(\sin x);$$

$$x = 0, \quad x = \pi$$

$$50 \quad y = 1 + 1.6x - 0.3x^2; \quad y = \sqrt{1 + x^3}$$

Exer 51–54: For each pair of net investment flows $I_1(t)$ and $I_2(t)$, (a) find the time interval during which I_1 is at least as great as I_2 , and (b) for the time interval found in part (a), determine how much more capital accumulates under the first investment flow than the second investment flow.

$$51 \quad I_1(t) = t; \quad I_2(t) = t^2$$

$$52 \quad I_1(t) = 4(1 - t^2); \quad I_2(t) = 1 - t^2$$

$$53 \quad I_1(t) = 2(1 - t^2); \quad I_2(t) = t^2 - 1$$

$$54 \quad I_1(t) = -t^2 + 4t; \quad I_2(t) = 3t/2$$

c **Exer. 55–56:** Graph, on the same coordinate axes, the given ellipses. (a) Estimate their points of intersection.

(b) Set up an integral that can be used to approximate the area of the region bounded by and inside both ellipses.

$$55 \quad \frac{x^2}{2.9} + \frac{y^2}{2.1} = 1; \quad \frac{x^2}{4.3} + \frac{(y - 2.1)^2}{4.9} = 1$$

$$56 \quad \frac{x^2}{3.9} + \frac{y^2}{2.4} = 1; \quad \frac{(x + 1.9)^2}{4.1} + \frac{y^2}{2.5} = 1$$

c **Exer. 57–58:** Graph, on the same coordinate axes, the given hyperbolas. (a) Estimate their first-quadrant point of intersection. (b) Set up an integral that can be used to approximate the area of the region in the first quadrant bounded by the hyperbolas and a coordinate axis.

$$57 \quad \frac{(y - 0.1)^2}{1.6} - \frac{(x + 0.2)^2}{0.5} = 1;$$

$$\frac{(y - 0.5)^2}{2.7} - \frac{(x - 0.1)^2}{5.3} = 1$$

$$58 \quad \frac{(x - 0.1)^2}{0.12} - \frac{y^2}{0.1} = 1; \quad \frac{x^2}{0.9} - \frac{(y - 0.3)^2}{2.1} = 1$$

5.2

SOLIDS OF REVOLUTION

The volume of an object plays an important role in many problems in the physical sciences. In this section and the next two sections, we consider several methods for computing volumes. Since it is difficult to determine the volume of an irregularly shaped object, we begin with objects that have simple shapes, including the solids of revolution.

If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**, and we say that the solid is **generated** by the region. The line is an **axis of revolution**. In particular, if the R_x region shown in Figure 5.20(a) is revolved about the x -axis, we obtain the solid illustrated in Figure 5.20(b). As a special case, if f is a constant function,

Figure 5.20

