

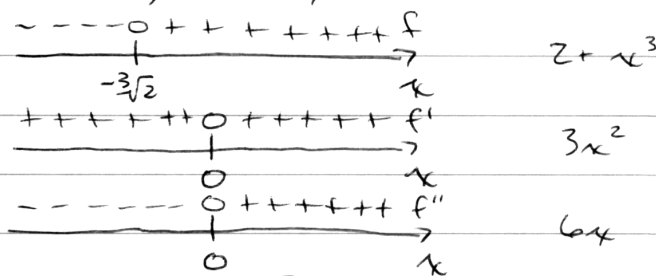
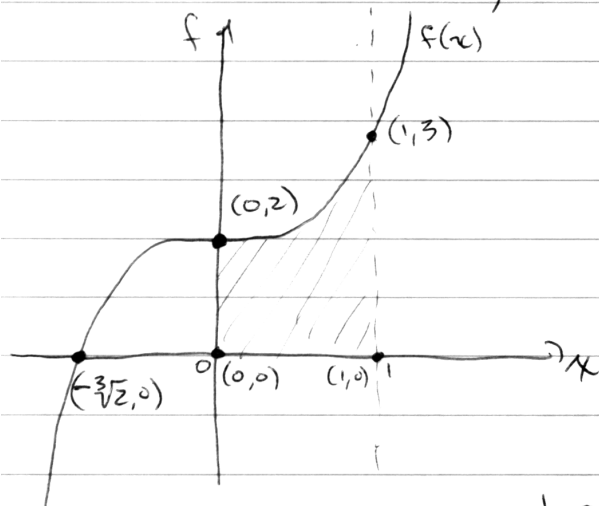
S.4: 1, 8, 16, 23

S.5: 1, 5, 8, 13, 14, 16, 19, 23, 25

B.4

In These problems, sketching a graph will be inordinately useful in solving the problems.

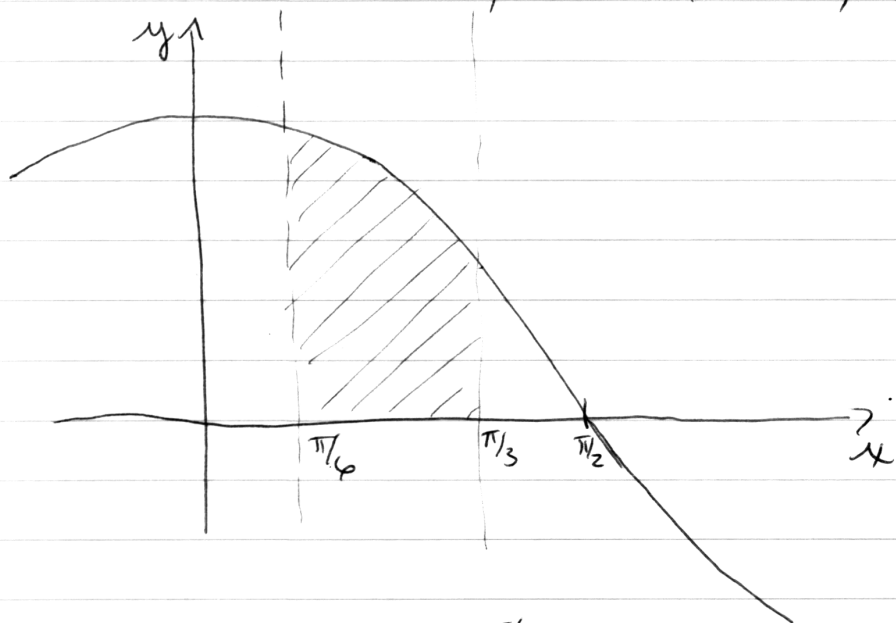
① Find the area of the region between the graph of $f(x) = 2 + x^3$, the x -axis, $x=0$, and $x=1$.



height = $f(x) - 0$
 " " " top - bottom of
 " " " the graph of
 width $x - 0$ the x -axis
 (NB: the x -axis is also known as the $y=0$ or the $g(x)=0$ line function)

$$\begin{aligned} \text{area desired} &= \int_0^1 (f(x) - 0) dx \\ &= \int_0^1 (2 + x^3) dx = \left[2x + \frac{x^4}{4} \right]_0^1 = (2 + 1/4) - (0 + 0) \\ &= 9/4 \end{aligned}$$

- ⑧ Find the area of the region bounded by the graph of $f(x) = \cos x$, the x -axis, $x = \pi/4$, and $x = \pi/3$.



$$\begin{aligned} \text{shaded area} &= \int_{\pi/4}^{\pi/3} \cos x \, dx \\ &= \sin x \Big|_{\pi/4}^{\pi/3} \end{aligned}$$

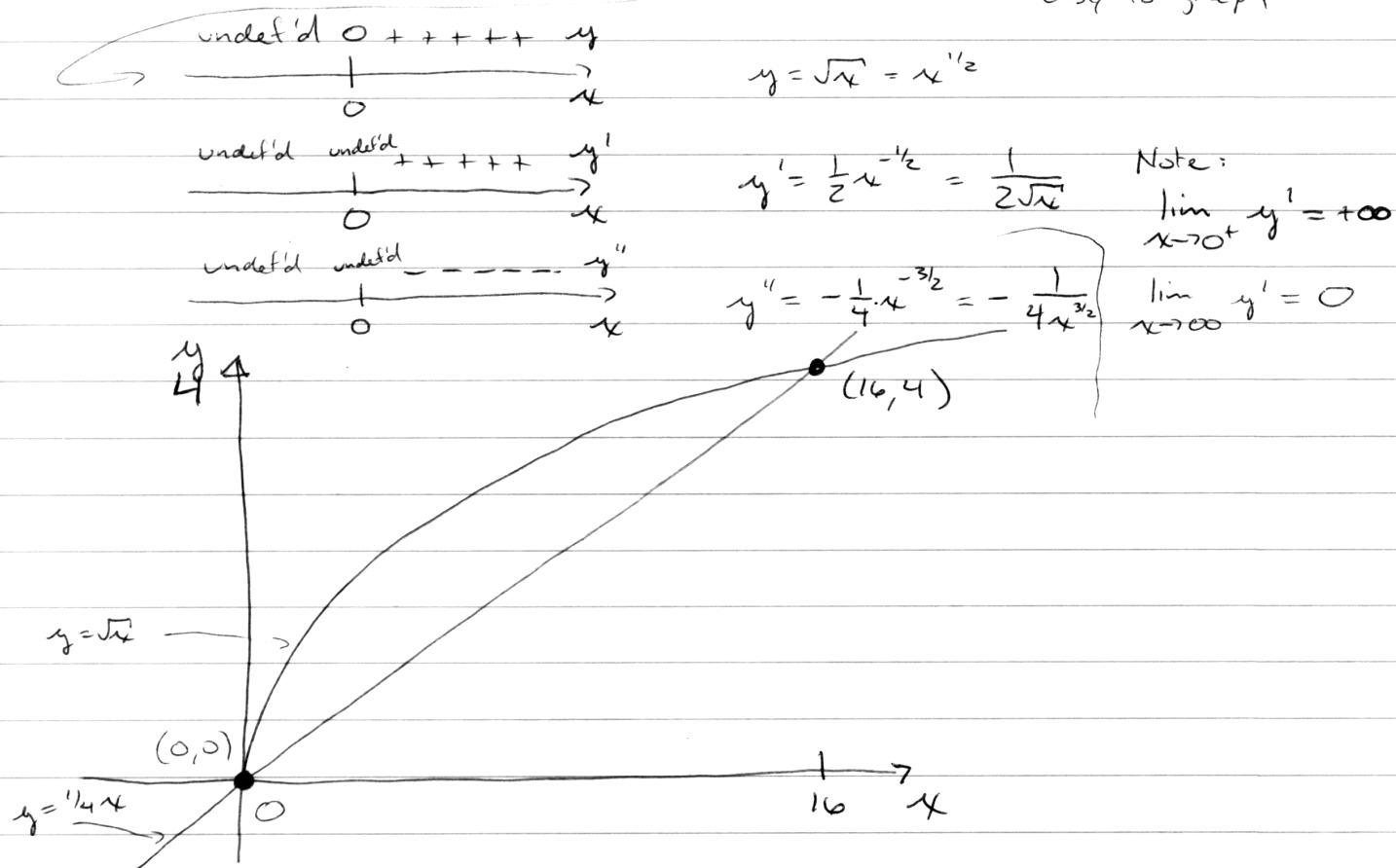
$$= \sin(\pi/3) - \sin(\pi/4)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$= \frac{1}{2}(\sqrt{3} - 1)$$

⑩ Find the area of the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{4}x$.

easy to graph

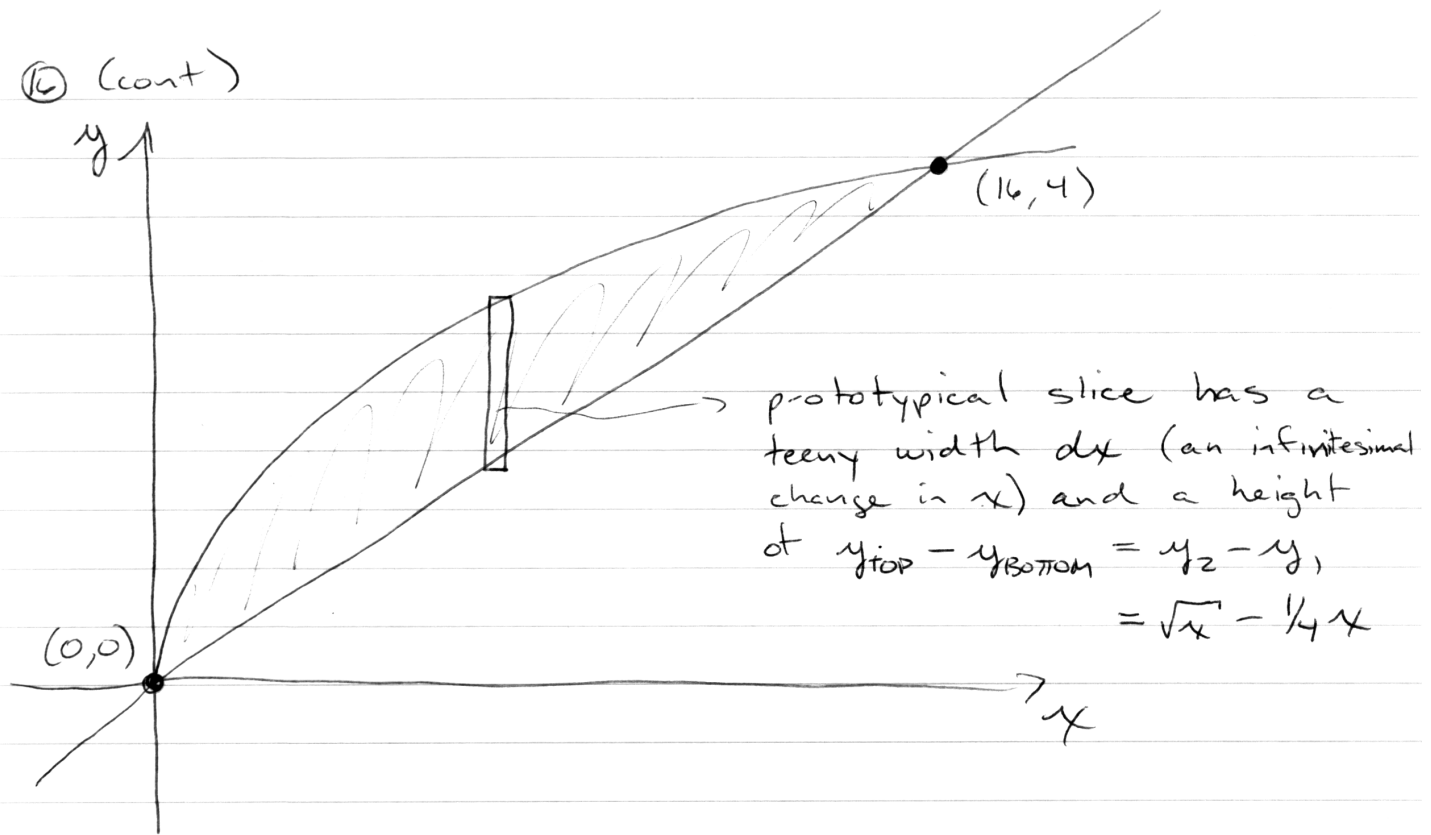


Curve ① $y_1 = \frac{1}{4}x_1$
 Curve ② $y_2 = \sqrt{x_2}$

Curves ① & ② intersect where $x_1 = x_2$ and $y_1 = y_2$ (that is, where the points on the curve coincide $(x_1, y_1) = (x_2, y_2)$).

$y_1 = y_2 \Rightarrow \frac{1}{4}x_1 = \sqrt{x_2}$ along with $x_1 = x_2$ this becomes $\frac{1}{4}x_1 = \sqrt{x_1} \Rightarrow \frac{1}{16}x_1^2 = x_1 \Rightarrow x_1(\frac{1}{16}x_1 - 1) = 0$
 $\Rightarrow x_1 = 0$ or $x_1 = 16$ (thus $x_2 = 0$ or $x_2 = 16$ and $y_1 = 0 = y_2$ or $y_1 = 4 = y_2$).

⑥ (cont)



shaded area = sum of ^{areas of} slices

$$= \int_{x=0}^{x=16} (\sqrt{x} - \frac{1}{4}x) dx$$

stop at the rightmost slice

start at the leftmost slice

$$= \left[\frac{2}{3} x^{3/2} - \frac{1}{8} x^2 \right]_0^{16}$$

$$= \left(\frac{2}{3} 16^{3/2} - \frac{1}{8} \cdot 16^2 \right) - (0 - 0)$$

$$= \frac{2}{3} \cdot 64 - \frac{1}{8} \cdot 256$$

$$= \frac{128}{3} - 32$$

$$= \frac{32}{3}$$

②③ Find the area of the region bounded by the graphs of $y = \cos x$ and $y = 4x^2 - \pi^2$.

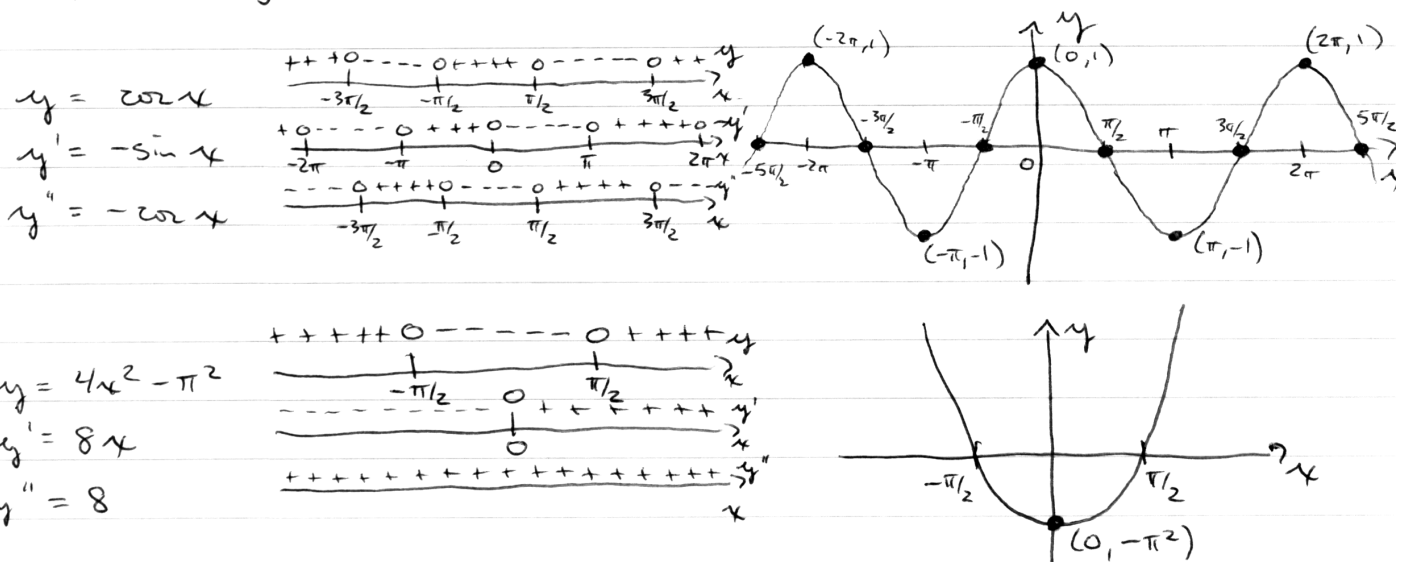
As in the last problem, we ultimately needed two things to find the area (along with some knowledge of integrals): where the two curves intersected, and which curve was the "top" and which was the "bottom" around the region of interest.

Starting on the first part, we want to find those x 's for which $\cos x = 4x^2 - \pi^2$.

We're stuck, though, because we don't have the tools to attack such an equation directly (try it).

We also similarly stuck to determine when $\cos x > 4x^2 - \pi^2$ and when $\cos x < 4x^2 - \pi^2$.

We can, however, try to draw a picture and hope things are more obvious.



From the graphs (and the analysis done to get them), it is clear that the only points of intersection of the

(23) (cont)

two graphs are $(-\pi/2, 0)$ and $(\pi/2, 0)$. On the interval $(-\pi/2, \pi/2)$, $\cos x > 0$ and $4x^2 - \pi^2 < 0$. Thus there can be no points of intersection on that interval (and $\cos x > 4x^2 - \pi^2$ on that interval). On the intervals $(-\pi/2, -3\pi/2)$ and $(\pi/2, 3\pi/2)$, $\cos x < 0$ and $4x^2 - \pi^2 > 0$. Thus there can be no points of intersection on these intervals. Also

$$\cos(-3\pi/2) = 0 < 8\pi^2 = 4(-3\pi/2)^2 - \pi^2 \text{ and}$$

$\cos(3\pi/2) = 0 < 8\pi^2 = 4(3\pi/2)^2 - \pi^2$ so the graphs do not intersect at $x = \pm 3\pi/2$. Also,

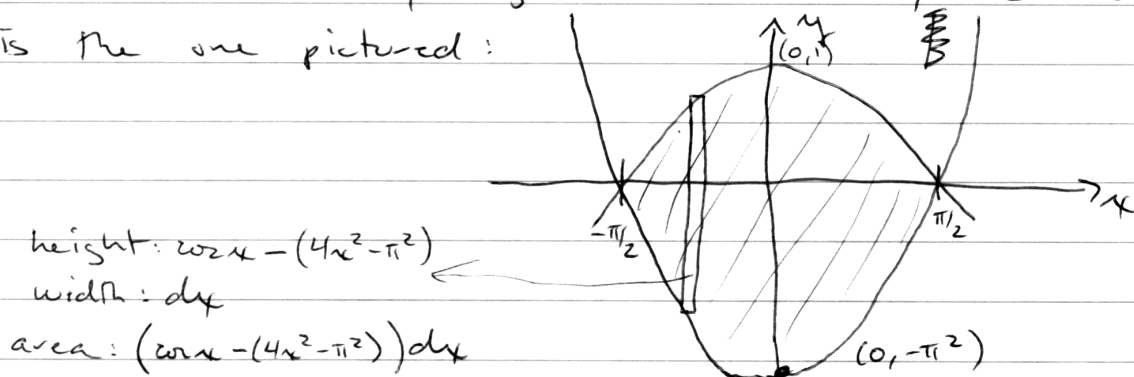
$$\cos x \leq 1 < 8\pi^2 \leq 4x^2 - \pi^2 \text{ for all } x \geq 3\pi/2$$

since $4x^2 - \pi^2$ is always increasing for those values of x (and $-1 \leq \cos x \leq 1$ for all x). Likewise,

$$\cos x \leq 1 < 8\pi^2 \leq 4x^2 - \pi^2 \text{ for all } x \leq -3\pi/2$$

since $4x^2 - \pi^2$ is always decreasing for those values of x . Thus there are no points of intersection on the intervals $(-\infty, -3\pi/2]$ and $[3\pi/2, +\infty)$ and on those intervals $\cos x < 4x^2 - \pi^2$.

Moreover the only region bounded by the two graphs is the one pictured:



(23) (cont)

The area is the sum of all the areas of the slices (a prototypical one is shown above).

$$\text{area desired} = \int_{-\pi/2}^{\pi/2} (\cos x - (4x^2 - \pi^2)) dx$$

$$= \left[\sin x - \frac{4x^3}{3} + \pi^2 x \right]_{-\pi/2}^{\pi/2}$$

$$= \left(\sin(\pi/2) - \frac{4}{3}\left(\frac{\pi}{2}\right)^3 + \pi^2\left(\frac{\pi}{2}\right) \right) -$$

$$\left(\sin(-\pi/2) - \frac{4}{3}\left(-\frac{\pi}{2}\right)^3 + \pi^2\left(-\frac{\pi}{2}\right) \right)$$

$$= \left(1 - \frac{\pi^3}{6} + \frac{\pi^3}{2} \right) - \left(-1 + \frac{\pi^3}{6} - \frac{\pi^3}{2} \right)$$

$$= 2 + \frac{2\pi^3}{3}$$

5.5

$$\textcircled{1} \int \frac{dx}{x^4} = \int x^{-4} dx$$

We know $\frac{d}{dx}(x^{-3}) = -3x^{-4}$. Thus $\frac{d}{dx}\left(\frac{x^{-3}}{-3}\right) = x^{-4}$.

$$\text{so that } \int x^{-4} dx = -\frac{1}{3}x^{-3} + C$$

$$\textcircled{5} \int \frac{dx}{\sqrt{1+x}} = \int (1+x)^{-1/2} dx$$

We know $\frac{d}{dx}((1+x)^{1/2}) = \frac{1}{2}(1+x)^{-1/2}(1) = \frac{1}{2}(1+x)^{-1/2}$

Thus $\frac{d}{dx}(2(1+x)^{1/2}) = (1+x)^{-1/2}$ so that $\int (1+x)^{-1/2} dx = 2(1+x)^{1/2} + C$

$$\textcircled{8} \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \int (x^{1/2} - x^{-1/2}) dx = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

$$\textcircled{13} \int g(x)g'(x) dx$$

We know $\frac{d}{dx}((g(x))^2) = 2g(x)g'(x)$ (think chain

rule; put \boxed{x} in place of $g(x)$ if you want). Thus

$\frac{d}{dx}\left(\frac{1}{2}(g(x))^2\right) = g(x)g'(x)$ so that $\int g(x)g'(x) dx = \frac{1}{2}(g(x))^2 + C$

$$\textcircled{14} \int \sin x \cos x dx = \frac{1}{2}(\sin x)^2 + C \text{ by the above rule.}$$

using $g(x) = \sin x$. (Note: This is the idea behind the

coming sections on u -substitutions. Let

⑭ (cont) $u = \sin x$

$$\frac{du}{dx} = \cos x \iff du = (\cos x) dx$$

$$\int \sin x \cos x dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\sin x)^2 + C$$

Is equivalent to:

$$g(x) = \sin x$$

$$g'(x) = \cos x \quad \left(\text{NB: } g'(x) = \frac{dg}{dx} = \cos x \iff dg = (\cos x) dx \right)$$

$$\int \sin x \cos x dx = \int g(x) g'(x) dx = \frac{1}{2} (g(x))^2 + C$$

$$= \frac{1}{2} (\sin x)^2 + C$$

Notwithstanding all the above, there are other ways to find an antiderivative for the above problem. One such way involves finding an alternate representation for the trigonometric integrand. One such representation is:

$$(\sin x)(\cos x) = \frac{1}{2} \sin(2x)$$

Thus

$$\int \sin x \cos x dx = \int \frac{1}{2} \sin(2x) dx = -\frac{1}{4} \cos(2x) + D$$

$$\left(\text{Note: } \frac{d}{dx} \left(-\frac{1}{4} \cos(2x) \right) = \left(-\frac{1}{4} \right) (-\sin(2x)) (2) = \frac{1}{2} \sin(2x) \right)$$

Although $\frac{1}{2} (\sin x)^2 \neq -\frac{1}{4} \cos(2x)$ for most all x , the

set of functions described by $\frac{1}{2} (\sin x)^2 + C$ is the same as that described by $-\frac{1}{4} \cos(2x) + D$ (Why? Can you figure it out?)

$$\textcircled{16} \int \frac{g'(x)}{(g(x))^2} dx = \int (g(x))^{-2} g'(x) dx = (-1)(g(x))^{-1} + C$$

$$\textcircled{19} \text{ Find } f(x) \text{ from } f'(x) = 2x - 1 \text{ \& } f(3) = 4$$

Since $\int (2x - 1) dx = x^2 - x + C$ we know that

our $f(x)$ is one of the functions from $x^2 - x + C$.

Since $f(3) = 4$, we must have $4 = 3^2 - 3 + C$ or

$C = -2$. Thus $f(x) = x^2 - x - 2$.

$$\textcircled{23} \text{ Find } f(x) \text{ from } f'(x) = \sin x \text{ and } f(0) = 2$$

Since $\int \sin x dx = -\cos x + C$ and $2 = -\cos(0) + C \Rightarrow$
 $C = 3,$

then $f(x) = -\cos x + 3$

$$\textcircled{25} \text{ Find } f(x) \text{ from } f''(x) = 6x - 2 \text{ \& } f'(0) = 1 \text{ \& } f(0) = 2$$

Since $\int (6x - 2) dx = 3x^2 - 2x + C$ and $1 = 3 \cdot 0^2 - 2 \cdot 0 + C$
 $\Rightarrow C = 1,$

then $f'(x) = 3x^2 - 2x + 1$.

Since $\int (3x^2 - 2x + 1) dx = x^3 - x^2 + x + D$ and
 $2 = 0^3 - 0^2 + 0 + D \Rightarrow D = 2,$ then

$f(x) = x^3 - x^2 + x + 2$