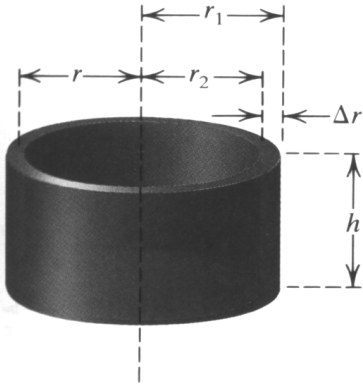


5.3 VOLUMES BY CYLINDRICAL SHELLS



Figure 5.31



In the preceding section, we found volumes of solids of revolution by using circular disks or washers. In this section, we shall see that for certain types of solids, it is convenient to use hollow circular cylinders—that is, thin **cylindrical shells** of the type illustrated in Figure 5.31, where r_1 is the *outer radius*, r_2 is the *inner radius*, h is the *altitude*, and $\Delta r = r_1 - r_2$ is the *thickness* of the shell. The **average radius** of the shell is $r = \frac{1}{2}(r_1 + r_2)$. We can find the volume of the shell by subtracting the volume $\pi r_2^2 h$ of the inner cylinder from the volume $\pi r_1^2 h$ of the outer cylinder. If we do so and change the form of the resulting expression, we obtain

$$\begin{aligned}\pi r_1^2 h - \pi r_2^2 h &= \pi(r_1^2 - r_2^2)h \\ &= \pi(r_1 + r_2)(r_1 - r_2)h \\ &= 2\pi \cdot \frac{1}{2}(r_1 + r_2)h(r_1 - r_2) \\ &= 2\pi r h \Delta r,\end{aligned}$$

which gives us the following general rule.

$$V = 2\pi(\text{average radius})(\text{altitude})(\text{thickness})$$

If the R_x region in Figure 5.32(a) is revolved about the y -axis, we obtain the solid illustrated in Figure 5.32(b).

Let P be a partition of $[a, b]$, and consider the typical vertical rectangle in Figure 5.32(c), where w_k is the midpoint of $[x_{k-1}, x_k]$. If we revolve this rectangle about the y -axis, we obtain a cylindrical shell of average radius w_k , altitude $f(w_k)$, and thickness Δx_k . Hence, by (5.10), the volume of the shell is

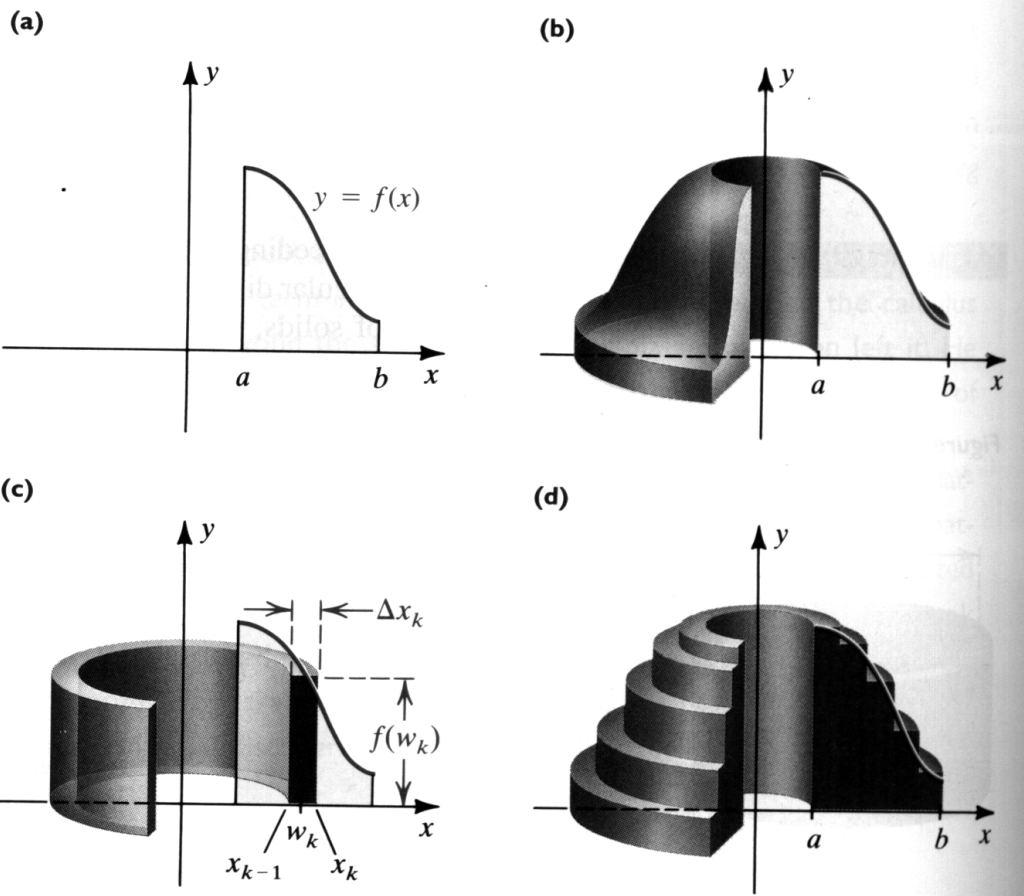
$$2\pi w_k f(w_k) \Delta x_k.$$

Revolving the rectangular polygon formed by *all* the rectangles determined by P gives us the solid illustrated in Figure 5.32(d). The volume of this solid is a Riemann sum:

$$\sum_k 2\pi w_k f(w_k) \Delta x_k$$

The smaller the norm $\|P\|$ of the partition, the better the sum approximates the volume V of the solid shown in Figure 5.32(b). This discussion provides the motivation for Definition (5.11) on the following page.

Figure 5.32



Definition 5.11

Let f be continuous and suppose $f(x) \geq 0$ on the interval $[a, b]$, where $0 \leq a \leq b$. Let R be the region under the graph of f from a to b . The volume V of the solid of revolution generated by revolving R about the y -axis is

$$V = \lim_{\|P\| \rightarrow 0} \sum_k 2\pi w_k f(w_k) \Delta x_k = \int_a^b 2\pi x f(x) dx.$$

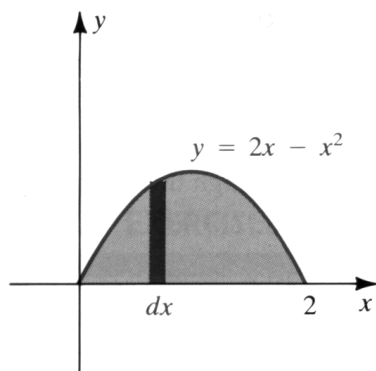
It can be proved that if the methods of Section 5.2 are also applicable, then both methods lead to the same answer.

We may also consider solids obtained by revolving a region about the x -axis or some other line. The following guidelines may be useful.

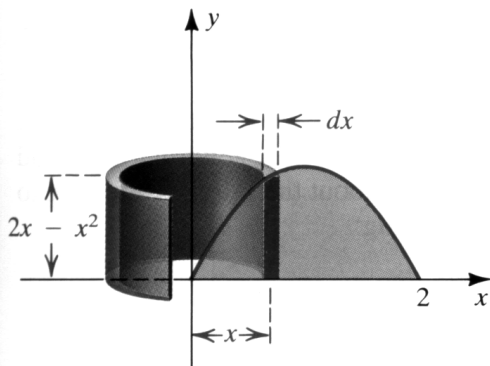
Guidelines for Finding the Volume of a Solid of Revolution Using Cylindrical Shells 5.12

- 1 Sketch the region R to be revolved, and label the boundaries. Show a typical vertical rectangle of width dx or a horizontal rectangle of width dy .
- 2 Sketch the cylindrical shell generated by the rectangle in guideline (1).
- 3 Express the average radius of the shell in terms of x or y , depending on whether its thickness is dx or dy . Remember that x represents a distance from the y -axis to a vertical rectangle, and y represents a distance from the x -axis to a horizontal rectangle.
- 4 Express the altitude of the shell in terms of x or y , depending on whether its thickness is dx or dy .
- 5 Use (5.10) to find a formula for the volume of the shell.
- 6 Apply the limit of sums operator \int_a^b or \int_c^d to the expression in guideline (5) and evaluate the integral.

Figure 5.33
(a)



(b)



EXAMPLE ■ I The region bounded by the graph of $y = 2x - x^2$ and the x -axis is revolved about the y -axis. Find the volume of the resulting solid.

SOLUTION The region to be revolved is sketched in Figure 5.33(a), together with a typical vertical rectangle of width dx . Figure 5.33(b) shows the cylindrical shell generated by revolving the rectangle about the y -axis. Note that x represents the distance from the y -axis to the midpoint of the rectangle (the average radius of the shell). Referring to the figure and using (5.10) gives us the following:

$$\begin{aligned} \text{thickness of shell:} & \quad dx \\ \text{average radius:} & \quad x \\ \text{altitude:} & \quad 2x - x^2 \\ \text{volume:} & \quad 2\pi x(2x - x^2) dx \end{aligned}$$

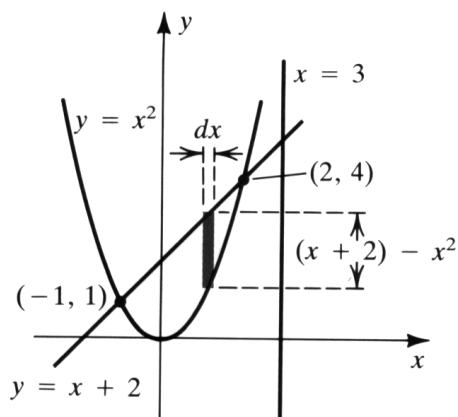
To sum all such shells, we move from left to right through the region from $a = 0$ to $b = 2$ (do *not* sum from -2 to 2). Hence, the limit of sums is

$$\begin{aligned} V &= \int_0^2 2\pi x(2x - x^2) dx = 2\pi \int_0^2 (2x^2 - x^3) dx \\ &= 2\pi \left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4} \right]_0^2 = \frac{8\pi}{3} \approx 8.4. \end{aligned}$$

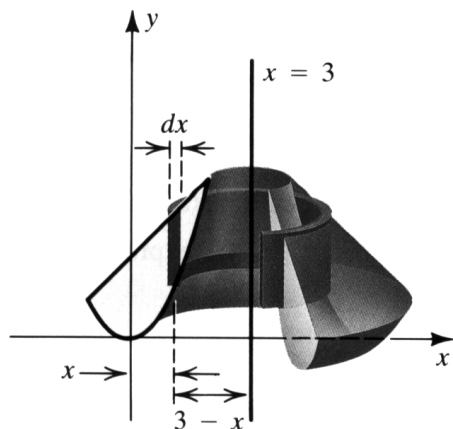
The volume V can also be found using washers; however, the calculations would be much more involved, since the equation $y = 2x - x^2$ would have to be solved for x in terms of y .

Figure 5.34

(a)



(b)



EXAMPLE ■ 2 The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Set up the integral for the volume of the resulting solid.

SOLUTION The region is sketched in Figure 5.34(a), together with a typical vertical rectangle extending from the lower boundary $y = x^2$ to the upper boundary $y = x + 2$. Also shown is the axis of revolution $x = 3$. In Figure 5.34(b), we have illustrated both the cylindrical shell and the solid that are generated by revolving the rectangle and the region about the line $x = 3$. It is important to note that since x is the distance from the y -axis to the rectangle, the radius of the shell is $3 - x$. Referring to Figure 5.34 and using (5.10) gives us the following:

$$\begin{aligned} \text{thickness of shell:} & \quad dx \\ \text{average radius:} & \quad 3 - x \\ \text{altitude:} & \quad (x + 2) - x^2 \\ \text{volume:} & \quad 2\pi(3 - x)(x + 2 - x^2) dx \end{aligned}$$

To sum all such shells, we move from left to right through the region from $a = -1$ to $b = 2$. Hence, the limit of sums is

$$V = \int_{-1}^2 2\pi(3 - x)(x + 2 - x^2) dx.$$

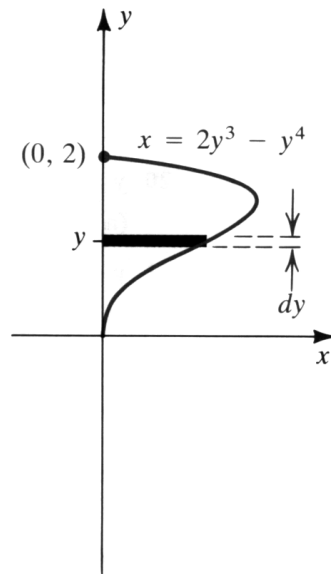
EXAMPLE ■ 3 The region in the first quadrant bounded by the graph of the equation $x = 2y^3 - y^4$ and the y -axis is revolved about the x -axis. Set up the integral for the volume of the resulting solid.

SOLUTION The region is sketched in Figure 5.35(a), together with a typical horizontal rectangle. Figure 5.35(b) shows the cylindrical shell and the solid that are generated by the revolution about the x -axis. Referring to the figure and using (5.10) gives the following:

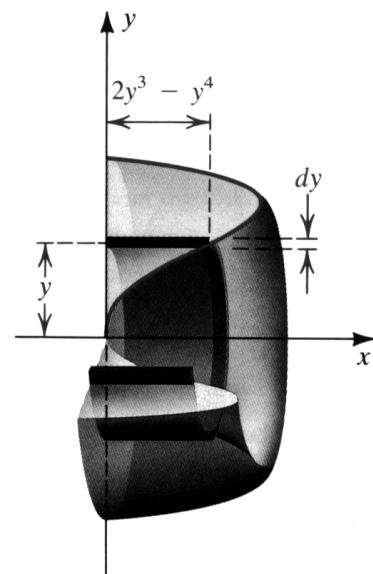
$$\begin{aligned} \text{thickness of shell:} & \quad dy \\ \text{average radius:} & \quad y \\ \text{altitude:} & \quad 2y^3 - y^4 \\ \text{volume:} & \quad 2\pi y(2y^3 - y^4) dy \end{aligned}$$

Figure 5.35

(a)



(b)



To sum all such shells, we move upward through the region from $c = 0$ to $d = 2$. Hence, the limit of sums is

$$V = \int_0^2 2\pi y(2y^3 - y^4) dy.$$

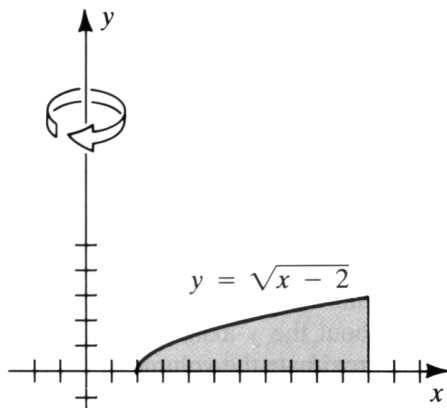
It is worth noting that in the preceding example we were forced to use shells and to integrate with respect to y , since the use of washers and integration with respect to x would require that we solve the equation $x = 2y^3 - y^4$ for y in terms of x , a rather formidable task.

EXERCISES 5.3

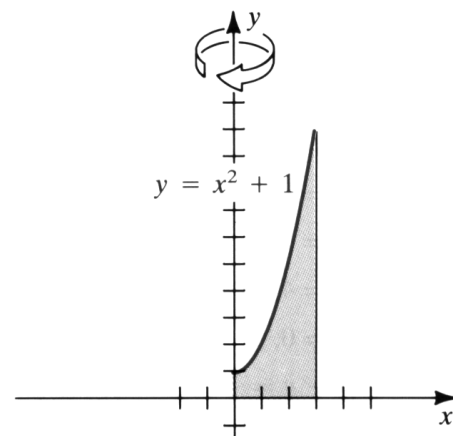
Use cylindrical shells for each exercise.

Exer. 1–4: Set up an integral that can be used to find the volume of the solid obtained by revolving the shaded region about the indicated axis.

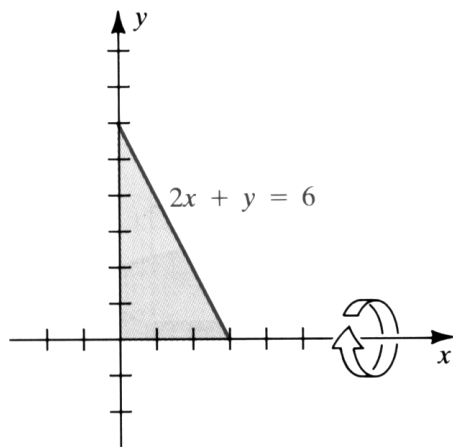
1



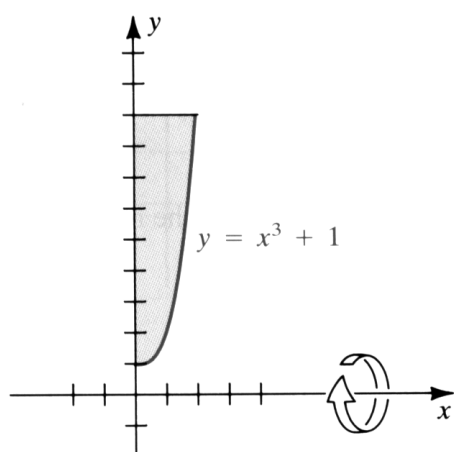
2



3



4



Exer. 5–18: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the indicated axis.

- | | | | | |
|----|----------------------|----------------|---------------------|-----------|
| 5 | $y = \sqrt{x}$, | $x = 4$, | $y = 0$; | y -axis |
| 6 | $y = 1/x$; | $x = 1$, | $x = 2$, $y = 0$; | y -axis |
| 7 | $y = x^2$, | $y^2 = 8x$; | | y -axis |
| 8 | $16y = x^2$, | $y^2 = 2x$; | | y -axis |
| 9 | $2x - y = 12$, | $x - 2y = 3$, | $x = 4$; | y -axis |
| 10 | $y = x^3 + 1$, | $x + 2y = 2$, | $x = 1$; | y -axis |
| 11 | $2x - y = 4$, | $x = 0$, | $y = 0$; | y -axis |
| 12 | $y = x^2 - 5x$, | $y = 0$; | | y -axis |
| 13 | $x^2 = 4y$, | $y = 4$; | | x -axis |
| 14 | $y^3 = x$, | $y = 3$, | $x = 0$; | x -axis |
| 15 | $y = 2x$, | $y = 6$, | $x = 0$; | x -axis |
| 16 | $2y = x$, | $y = 4$, | $x = 1$; | x -axis |
| 17 | $y = \sqrt{x + 4}$, | $y = 0$, | $x = 0$; | x -axis |
| 18 | $y = -x$, | $x - y = -4$, | $y = 0$; | x -axis |

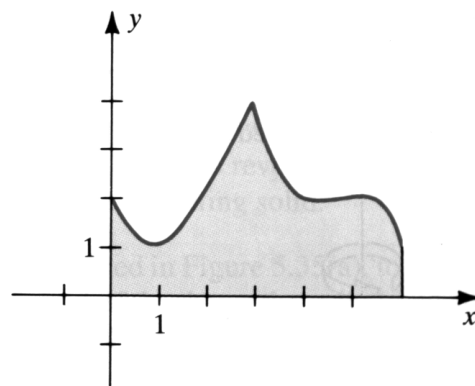
Exer. 19–26: Let R be the region bounded by the graphs of the equations. Set up an integral that can be used to find the volume of the solid generated if R is revolved about the given line.

- 19 $y = x^2 + 1$, $x = 0$, $x = 2$, $y = 0$
 (a) $x = 3$ (b) $x = -1$
- 20 $y = 4 - x^2$, $y = 0$
 (a) $x = 2$ (b) $x = -3$
- 21 $y = x^2$, $y = 4$
 (a) $y = 4$ (b) $y = 5$ (c) $x = 2$ (d) $x = -3$
- 22 $y = \sqrt{x}$, $y = 0$, $x = 4$
 (a) $x = 4$ (b) $x = 6$ (c) $y = 2$ (d) $y = -4$
- 23 $x + y = 3$, $y + x^2 = 3$; $x = 2$
- 24 $y = 1 - x^2$, $x - y = 1$; $y = 3$
- 25 $x^2 + y^2 = 1$; $x = 5$
- 26 $y = x^{2/3}$, $y = x^2$; $y = -1$

Exer. 27–30: Let R be the region bounded by the graphs of the equations. Set up integrals that can be used to find the volume of the solid generated if R is revolved about the given axis using (a) cylindrical shells and (b) disks or washers.

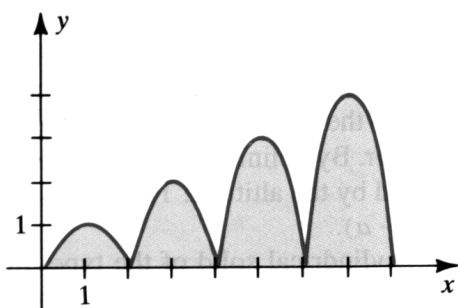
- 27 $y = 1/\sqrt{x}$, $x = 1$, $x = 4$, $y = 0$; x -axis
- 28 $y = 9 - x^2$, $x = 0$, $x = 2$, $y = 0$; x -axis
- 29 $y = x^2 + 2$, $x = 0$, $x = 1$, $y = 0$; y -axis
- 30 $y = x + 1$, $x = 0$, $x = 1$, $y = 0$; y -axis
- 31 If the region shown in the figure is revolved about the y -axis, use the trapezoidal rule, with $n = 6$, to approximate the volume of the resulting solid.

Exercise 31



- 32 If the region shown in the figure on the following page is revolved about the y -axis, use Simpson's rule, with $n = 4$, to approximate the volume of the resulting solid.

Exercise 32



c 33 Graph $f(x) = -x^4 + 2.21x^3 - 3.21x^2 + 4.42x - 2$.

- Estimate the x -intercepts of the graph.
- If the region bounded by the graph of f and the x -axis is revolved about the y -axis, set up an integral that can be used to approximate the volume of the resulting solid.

c 34 Graph, on the same coordinate axes, $f(x) = \csc x$ and $g(x) = x + 1$ for $0 < x < \pi$.

- Use Newton's method to approximate, to four decimal places, the x -coordinates of the points of intersection of the graphs.
- If the region bounded by the graphs is revolved

about the y -axis, use the trapezoidal rule, with $n = 6$, to approximate the volume of the resulting solid.

35 Let R be the region bounded by the parabola $x^2 = 4y$ and the line l through the focus that is perpendicular to the axis of the parabola.

- Find the area of R .
- If R is revolved about the y -axis, find the volume of the resulting solid.
- If R is revolved about the x -axis, find the volume of the resulting solid.

36 Work (a)–(c) of Exercise 35 if R is the region bounded by the graphs of $y^2 = 2x - 6$ and $x = 5$.

Exer. 37–38: Let R be the region bounded by the hyperbola with equation $b^2x^2 - a^2y^2 = a^2b^2$ and a vertical line through a focus.

37 Show that the area of the region R is given by

$$\frac{2b}{a} \int_a^c \sqrt{x^2 - a^2} dx, \text{ where } c = \sqrt{a^2 + b^2}.$$

38 Find the volume of the solid obtained by revolving R about the y -axis.

5.4

VOLUMES BY CROSS SECTIONS



If a plane intersects a solid, then the region common to the plane and the solid is a **cross section** of the solid. In Section 5.2, we used circular and washer-shaped cross sections to find volumes of solids of revolution. In this section, we shall study solids that have the following property (see Figure 5.36): For every x in $[a, b]$, the plane perpendicular to the x -axis at x intersects the solid in a cross section whose area is $A(x)$, where A is a continuous function on $[a, b]$.

Figure 5.36

