

7.3 For #1, 4, 10, find the domain and the derivative of the given function.

① $f(x) = \log(4x)$

The argument to a logarithm must be positive for the log to be defined. Thus we need $4x > 0$ or $x > 0$. Thus the domain of f is $(0, \infty)$.

$$f'(x) = \frac{d}{dx}(\log(4x)) = \frac{1}{4x} \cdot \frac{d}{dx}(4x) = \frac{1}{4x} \cdot 4 = \frac{1}{x}$$

④ $f(x) = \log((x+1)^3)$

Again we need $(x+1)^3 > 0$. Since cube roots preserve sign, this inequality is the same as $x+1 > 0$ or $x > -1$. The domain of f is $(-1, \infty)$.

$$f'(x) = \frac{1}{(x+1)^3} \cdot \frac{d}{dx}((x+1)^3) = \frac{1}{(x+1)^3} \cdot 3(x+1)^2(1)$$

Since f is defined only for $x > -1$, at best f' is defined for $x > -1$. Since for $x > -1$

$$\frac{(x+1)^2}{(x+1)^3} = \frac{1}{x+1}, \quad f'(x) = \frac{3}{x+1}.$$

⑩ $f(x) = \log\left(\left|\frac{x+2}{x^3-1}\right|\right)$

We need the argument to log to be positive. Since the absolute value of any number is non-negative, the only time the log will be undefined is when $\frac{x+2}{x^3-1} = 0$.

⑩ (cont)

This only happens when $x = -2$. There is the additional problem that $\frac{x+2}{x^3-1}$ is itself undefined for $x=1$. Thus f has a domain of all real numbers except -2 and 1 (that is, $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$).

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\log \left(\left| \frac{x+2}{x^3-1} \right| \right) \right) = \frac{1}{\frac{x+2}{x^3-1}} \cdot \frac{d}{dx} \left(\frac{x+2}{x^3-1} \right) \\ &= \frac{x^3-1}{x+2} \cdot \frac{(1)(x^3-1) - (x+2)(3x^2)}{(x^3-1)^2} = \frac{x^3-1}{x+2} \cdot \frac{-2x^3-6x^2-1}{(x^3-1)^2} \\ &= \frac{-2x^3-6x^2-1}{(x+2)(x^3-1)} \end{aligned}$$

(Note: all the above steps are valid since we're assuming $x \neq -2$ & $x \neq 1$; also, $2x^3 + 6x^2 + 1$ has no rational roots.)

⑮ $\int \frac{dx}{x+1} = \int \frac{du}{u} = \log(|u|) + C = \log(|x+1|) + C$

if $u = x+1$
and $du = dx$

⑰ $\int \frac{x dx}{3-x^2} = \int \frac{-1/2 du}{u} = -\frac{1}{2} \log(|u|) + C = -\frac{1}{2} \log(|3-x^2|) + C$

if $u = 3-x^2$
 $du = -2x dx$

(25)

$$\int \frac{\sin x \, dx}{2 + \cos x} = \int \frac{-du}{u} = -\log(|u|) + C = -\log(|2 + \cos x|) + C$$

$$\text{if } u = 2 + \cos x$$

$$du = -\sin x \, dx$$

(28)

$$\int \frac{x^2 \, dx}{2x^3 - 1} = \int \frac{\frac{1}{6} du}{u} = \frac{1}{6} \log(|u|) + C = \frac{1}{6} \log(|2x^3 - 1|) + C$$

$$\text{if } u = 2x^3 - 1$$

$$du = 6x^2 \, dx$$

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$$\int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{x+2}$$

$$= \int_1^2 \frac{du}{u} - \int_2^3 \frac{dv}{v}$$

$$u = x+1$$

$$du = dx$$

$$v = x+2$$

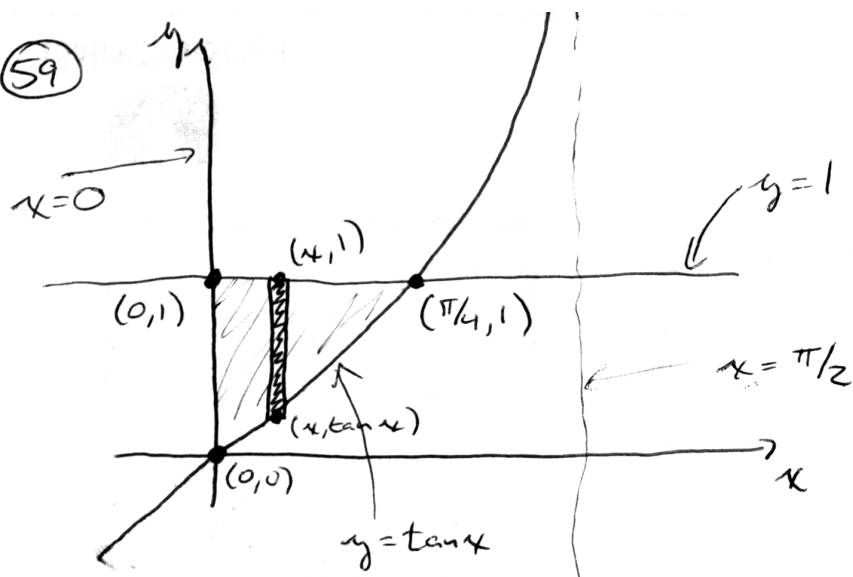
$$dv = dx$$

$$= \log(|u|) \Big|_1^2 - \log(|v|) \Big|_2^3$$

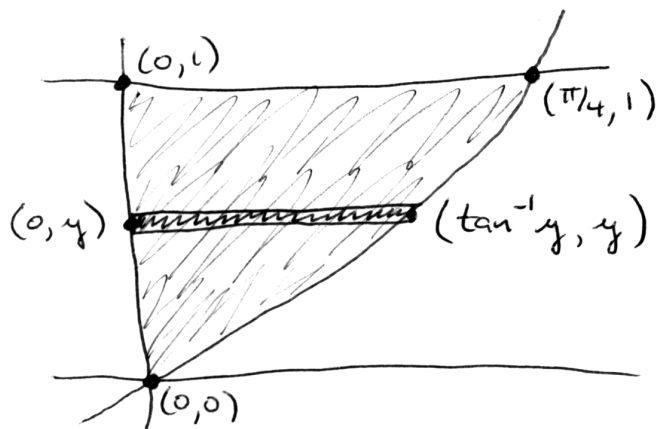
$$= (\log 2 - \log 1) - (\log 3 - \log 2)$$

$$= 2 \log 2 - \log 3$$

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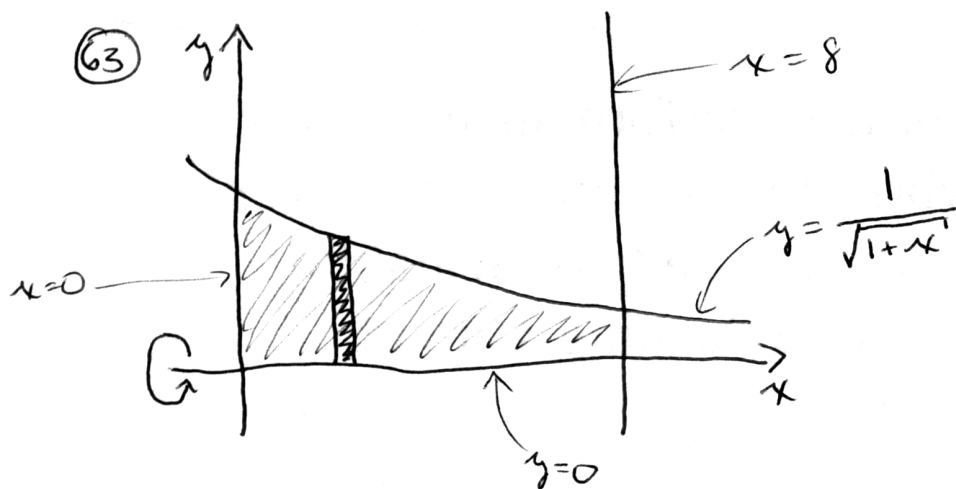


$$\begin{aligned}
 \text{area} &= \int_0^{\pi/4} (1 - \tan x) dx \\
 &= \left[x - \log(|\sec x|) \right]_0^{\pi/4} \\
 &= \left(\frac{\pi}{4} - \log(\sqrt{2}) \right) - (0 - \log 1) \\
 &= \frac{\pi}{4} - \frac{1}{2} \log 2
 \end{aligned}$$



$$\begin{aligned}
 \text{area} &= \int_0^1 (\tan^{-1} y - 0) dy \\
 &= \int_0^1 \tan^{-1} y dy \\
 &= \left[y \tan^{-1} y - \frac{1}{2} \log(1+y^2) \right]_0^1 \\
 &= \left(1 \cdot \frac{\pi}{4} - \frac{1}{2} \log 2 \right) - \left(0 - \frac{1}{2} \log 1 \right) \\
 &= \frac{\pi}{4} - \frac{1}{2} \log 2
 \end{aligned}$$

Note; you probably don't know how to make this step yet

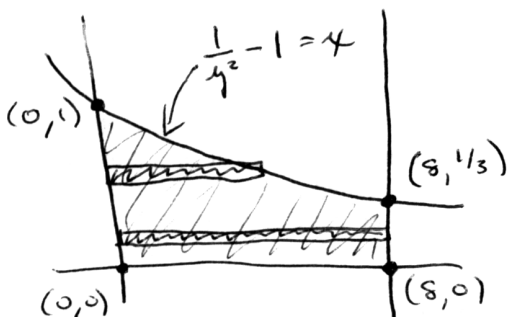


vol of a disc = $\left(\pi \left(\frac{1}{\sqrt{x+1}} \right)^2 \right) dx$

total volume = $\int_0^8 \frac{\pi dx}{x+1} = \int_1^9 \frac{\pi du}{u} = \pi \log(|u|) \Big|_1^9$

$= \pi \log 9 - \pi \log 1$

$= 2\pi \log 3$



vol of a shell =

$= \begin{cases} (2\pi y) \left(\frac{1}{y^2} - 1 \right) (dy) & \frac{1}{3} \leq y \leq 1 \\ (2\pi y) (8) (dy) & 0 \leq y \leq \frac{1}{3} \end{cases}$

total volume = $\int_0^{1/3} 16\pi y dy + \int_{1/3}^1 2\pi \left(\frac{1}{y} - y \right) dy$

$= 8\pi y^2 \Big|_0^{1/3} + 2\pi \left(\log(|y|) - \frac{1}{2} y^2 \right) \Big|_{1/3}^1$

$= \frac{8}{9} \pi + 2\pi \left(\left(0 - \frac{1}{2} \right) - \left(\log \frac{1}{3} - \frac{1}{18} \right) \right)$

$= 2\pi \log 3$

7.4 #1, 3, 17, 22: Differentiate the following functions.

① $y = e^{-2x}$

$$y' = \frac{d}{dx}(e^{-2x})$$
$$= e^{-2x} \cdot \frac{d}{dx}(-2x)$$

$$= -2e^{-2x}$$

③ $y = e^{x^2-1}$

$$y' = (e^{x^2-1}) \cdot \frac{d}{dx}(x^2-1)$$
$$= 2xe^{x^2-1}$$

⑦ $y = \frac{e^x - 1}{e^x + 1}$

$$y' = \frac{(e^x)(e^x + 1) - (e^x - 1)(e^x)}{(e^x + 1)^2}$$

$$= \frac{2e^x}{(e^x + 1)^2}$$

⑫ $y = e^{\sin 2x}$

$$y' = (e^{\sin 2x})(\cos 2x)(2)$$

#25, 29, 35, 37: Calculate the following indefinite integrals

$$\textcircled{25} \int e^{2x} dx = \int (e^u) \left(\frac{1}{2} du \right) = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x} + C$$

$$\begin{aligned} \text{if } u &= 2x \\ du &= 2 dx \end{aligned}$$

$\textcircled{29}$

$$\int x e^{x^2} dx = \int (e^u) \left(\frac{1}{2} du \right) = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

$$\begin{aligned} \text{if } u &= x^2 \\ du &= 2x dx \end{aligned}$$

$\textcircled{35}$

$$\int \frac{4}{\sqrt{e^x}} dx = \int 4 e^{-\frac{1}{2}x} dx = \int (4 e^u) (-2 du) = -8 e^u + C$$

$$\begin{aligned} u &= -\frac{1}{2}x \\ du &= -\frac{1}{2} dx \end{aligned} \qquad = -8 e^{-\frac{1}{2}x} + C$$

$\textcircled{37}$

$$\int \frac{e^x dx}{\sqrt{e^x + 1}} = \int \frac{du}{\sqrt{u}} = 2 u^{1/2} + C = 2 \sqrt{e^x + 1} + C$$

$$\begin{aligned} u &= e^x + 1 \\ du &= e^x dx \end{aligned}$$

$\textcircled{53}$

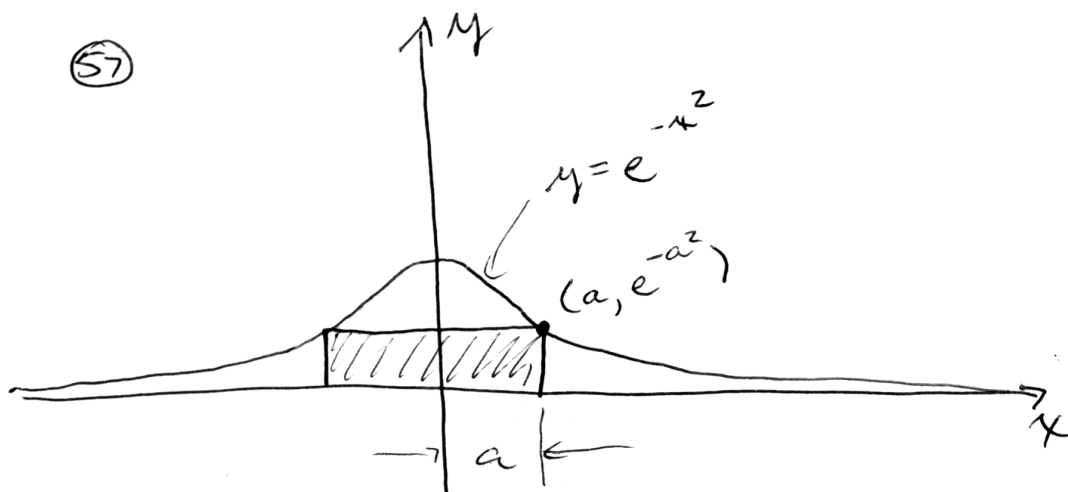
$$\int_0^1 x(e^{x^2} + 2) dx = \int_0^1 (e^u + 2) \left(\frac{1}{2} du \right) = \frac{1}{2} (e^u + 2u) \Big|_0^1$$

$$\begin{aligned} u &= x^2 \\ du &= 2x dx \end{aligned}$$

$$\begin{aligned} 0 &= 0^2 \\ 1 &= 1^2 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} [(e+2) - (1+0)] \\ &= \frac{1}{2} (e+1) \end{aligned}$$

(57)



Suppose $a \geq 0$. The rectangle is $2a$ wide. It's e^{-a^2} high and has an area of $2ae^{-a^2}$.

Let $A = 2ae^{-a^2}$ for $a \geq 0$. We want to maximize A over all $a (\geq 0)$. We can differentiate A wrt. a and find critical points (A is a smooth function of a). The maximum will lie at one of these points.

$$A = 2ae^{-a^2} \quad \text{for } a \geq 0$$

$$\frac{dA}{da} = (2)e^{-a^2} + (2a)(-2ae^{-a^2}) \quad \text{for } a > 0$$

There's a critical point at $a=0$ (where A is def'd but A' isn't) and for whatever a makes

$$2e^{-a^2} - 4a^2e^{-a^2} = 0 \quad \text{or} \quad \underbrace{(2e^{-a^2})}_{\text{never } 0} \underbrace{(1 - 2a^2)}_{0 \text{ when } a = 1/\sqrt{2}} = 0$$

$$\begin{array}{c} \text{+++++} \quad \text{-----} \quad A' \\ | \quad \quad \quad | \quad \quad \quad \rightarrow a \\ 0 \quad \quad 1/\sqrt{2} \end{array} \Rightarrow a = 1/\sqrt{2} \text{ is at a maximum of } A$$

Therefore the vertices being placed at $(1/\sqrt{2}, e^{-1/2})$ & $(-1/\sqrt{2}, e^{-1/2})$ will maximize the area of the specified rectangle