

$$48 \quad y = \sin[\pi(x^2 - 1)]; \quad y = 1 - x^2$$

$$49 \quad y = \sin x; \quad y = \sin(\sin x);$$

$$x = 0, \quad x = \pi$$

$$50 \quad y = 1 + 1.6x - 0.3x^2; \quad y = \sqrt{1 + x^3}$$

**Exer 51–54:** For each pair of net investment flows  $I_1(t)$  and  $I_2(t)$ , (a) find the time interval during which  $I_1$  is at least as great as  $I_2$ , and (b) for the time interval found in part (a), determine how much more capital accumulates under the first investment flow than the second investment flow.

$$51 \quad I_1(t) = t; \quad I_2(t) = t^2$$

$$52 \quad I_1(t) = 4(1 - t^2); \quad I_2(t) = 1 - t^2$$

$$53 \quad I_1(t) = 2(1 - t^2); \quad I_2(t) = t^2 - 1$$

$$54 \quad I_1(t) = -t^2 + 4t; \quad I_2(t) = 3t/2$$

**c** **Exer. 55–56:** Graph, on the same coordinate axes, the given ellipses. (a) Estimate their points of intersection.

**(b)** Set up an integral that can be used to approximate the area of the region bounded by and inside both ellipses.

$$55 \quad \frac{x^2}{2.9} + \frac{y^2}{2.1} = 1; \quad \frac{x^2}{4.3} + \frac{(y - 2.1)^2}{4.9} = 1$$

$$56 \quad \frac{x^2}{3.9} + \frac{y^2}{2.4} = 1; \quad \frac{(x + 1.9)^2}{4.1} + \frac{y^2}{2.5} = 1$$

**c** **Exer. 57–58:** Graph, on the same coordinate axes, the given hyperbolas. (a) Estimate their first-quadrant point of intersection. (b) Set up an integral that can be used to approximate the area of the region in the first quadrant bounded by the hyperbolas and a coordinate axis.

$$57 \quad \frac{(y - 0.1)^2}{1.6} - \frac{(x + 0.2)^2}{0.5} = 1;$$

$$\frac{(y - 0.5)^2}{2.7} - \frac{(x - 0.1)^2}{5.3} = 1$$

$$58 \quad \frac{(x - 0.1)^2}{0.12} - \frac{y^2}{0.1} = 1; \quad \frac{x^2}{0.9} - \frac{(y - 0.3)^2}{2.1} = 1$$

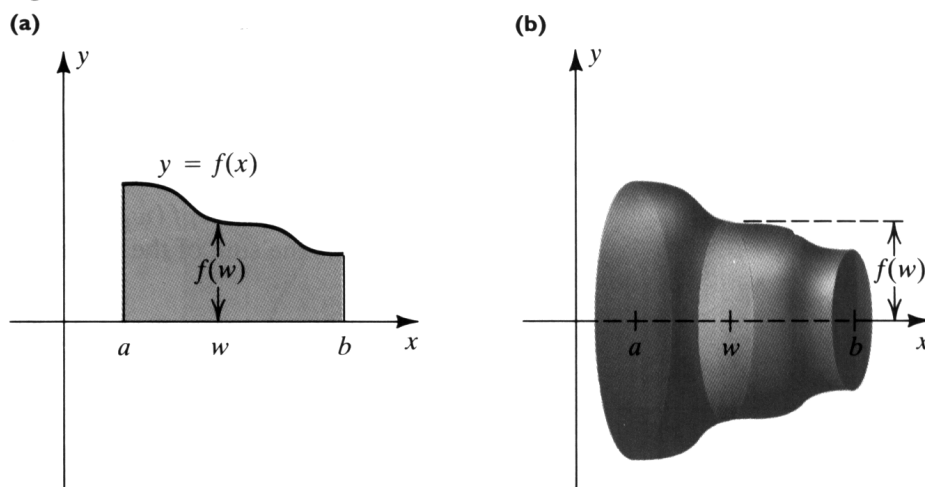
## 5.2

## SOLIDS OF REVOLUTION

The volume of an object plays an important role in many problems in the physical sciences. In this section and the next two sections, we consider several methods for computing volumes. Since it is difficult to determine the volume of an irregularly shaped object, we begin with objects that have simple shapes, including the solids of revolution.

If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**, and we say that the solid is **generated** by the region. The line is an **axis of revolution**. In particular, if the  $R_x$  region shown in Figure 5.20(a) is revolved about the  $x$ -axis, we obtain the solid illustrated in Figure 5.20(b). As a special case, if  $f$  is a constant function,

**Figure 5.20**

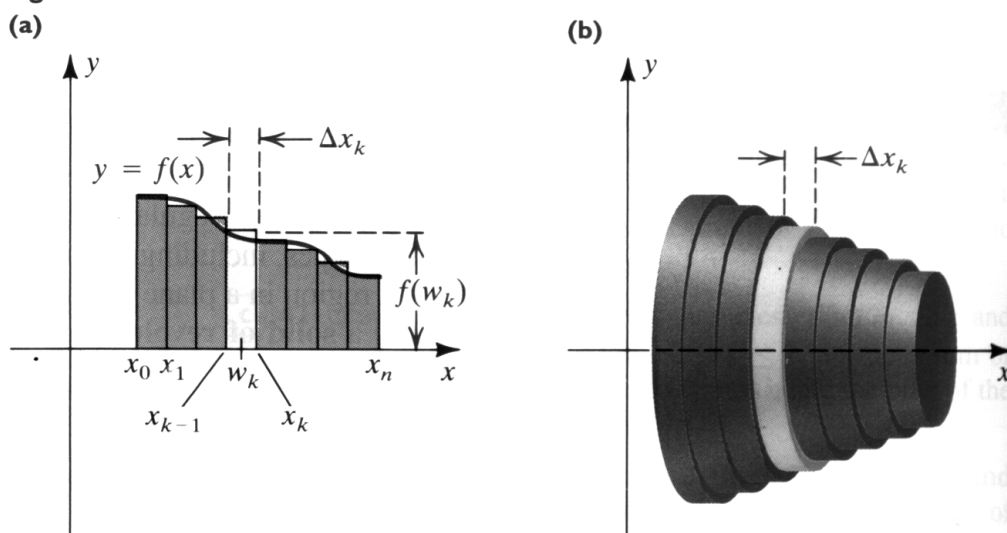


say  $f(x) = k$ , then the region is rectangular and the solid generated is a right circular cylinder. If the graph of  $f$  is a semicircle with endpoints of a diameter at the points  $(a, 0)$  and  $(b, 0)$ , then the solid of revolution is a sphere. If the region is a right triangle with base on the  $x$ -axis and two vertices at the points  $(a, 0)$  and  $(b, 0)$  with the right angle at one of these points, then the solid generated is a right circular cone.

If a plane perpendicular to the  $x$ -axis intersects the solid shown in Figure 5.20(b), a circular cross section is obtained. If, as indicated in the figure, the plane passes through the point on the axis with  $x$ -coordinate  $w$ , then the radius of the circle is  $f(w)$ , and hence its area is  $\pi[f(w)]^2$ . We shall arrive at a definition for the volume of such a solid of revolution by using Riemann sums.

Let us partition the interval  $[a, b]$ , as we did for areas in Section 5.1, and consider the rectangles in Figure 5.21(a). The solid of revolution generated by these rectangles has the shape shown in Figure 5.21(b). Beginning with Figure 5.25, we shall remove, or cut out, parts of solids of revolution to help us visualize portions generated by typical rectangles. When referring to such figures, remember that the entire solid is obtained by one *complete* revolution about an axis, not a partial one.

**Figure 5.21**



Observe that the  $k$ th rectangle generates a **circular disk** (a flat right circular cylinder) of base radius  $f(w_k)$  and altitude (thickness)  $\Delta x_k = x_k - x_{k-1}$ . The volume of this disk is the area of the base times the altitude—that is,  $\pi[f(w_k)]^2 \Delta x_k$ . The volume of the solid shown in Figure 5.21(b) is the sum of the volumes of all such disks:

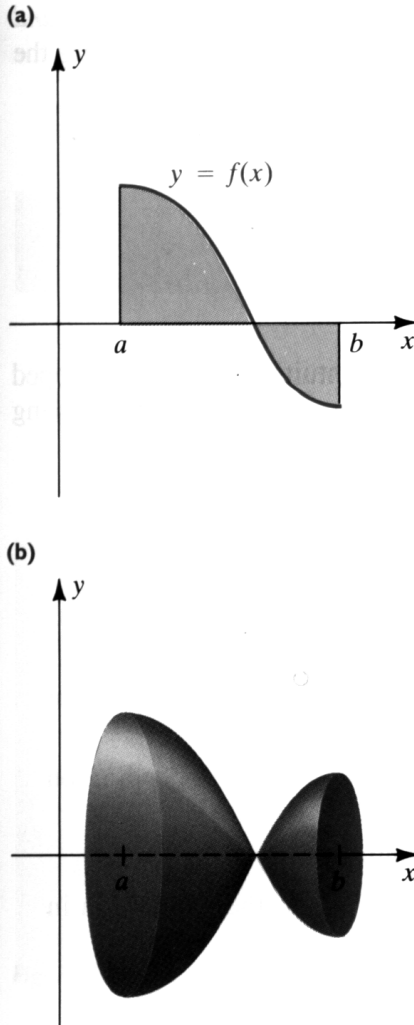
$$\sum_k \pi[f(w_k)]^2 \Delta x_k$$

This sum may be regarded as a Riemann sum for  $\pi[f(x)]^2$ . If the norm  $\|P\|$  of the partition is close to zero, then the sum should be close to the

volume of the solid. Hence we define the volume of the solid of revolution as a limit of these sums.

### Definition 5.5

Figure 5.22



Let  $f$  be continuous on  $[a, b]$ , and let  $R$  be the region bounded by the graph of  $f$ , the  $x$ -axis, and the vertical lines  $x = a$  and  $x = b$ . The **volume**  $V$  of the solid of revolution generated by revolving  $R$  about the  $x$ -axis is

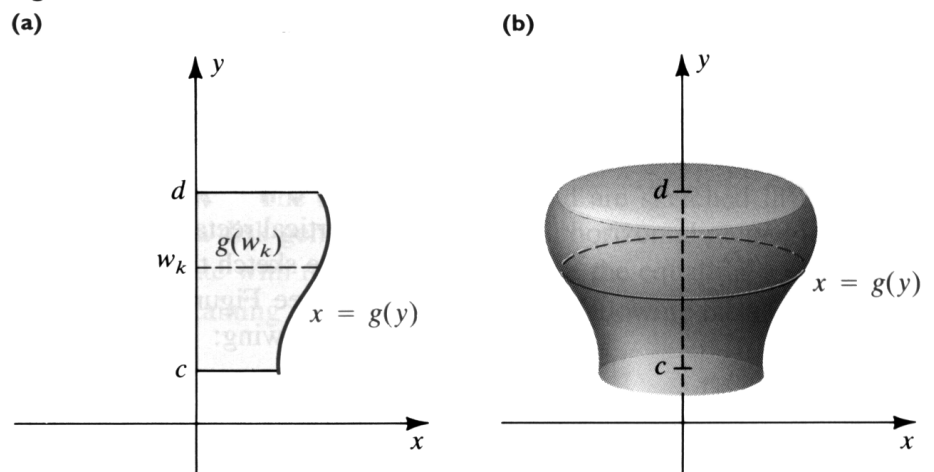
$$V = \lim_{\|P\| \rightarrow 0} \sum_k \pi [f(w_k)]^2 \Delta x_k = \int_a^b \pi [f(x)]^2 dx.$$

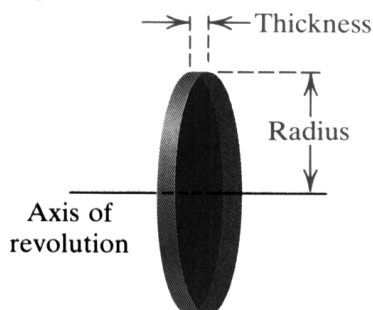
The fact that the limit of sums in this definition equals  $\int_a^b \pi [f(x)]^2 dx$  follows from the definition of the definite integral. We shall not ordinarily specify the units of measure for volume. If the linear measurement is inches, the volume is in cubic inches ( $\text{in}^3$ ). If  $x$  is measured in centimeters, then  $V$  is in cubic centimeters ( $\text{cm}^3$ ), and so on.

The requirement that  $f(x) \geq 0$  was omitted intentionally in Definition (5.5). If  $f$  is negative for some  $x$ , as in Figure 5.22(a), and if the region bounded by the graphs of  $f$ ,  $x = a$ ,  $x = b$ , and the  $x$ -axis is revolved about the  $x$ -axis, we obtain the solid shown in Figure 5.22(b). This solid is the same as that generated by revolving the region under the graph of  $y = |f(x)|$  from  $a$  to  $b$  about the  $x$ -axis. Since  $|f(x)|^2 = [f(x)]^2$ , the limit in Definition (5.5) gives us the volume.

Let us interchange the roles of  $x$  and  $y$  and revolve the  $R_y$  region in Figure 5.23(a) about the  $y$ -axis, obtaining the solid illustrated in Figure 5.23(b). If we partition the  $y$ -interval  $[c, d]$  and use *horizontal* rectangles of width  $\Delta y_k$  and length  $g(w_k)$ , the same type of reasoning that gave us (5.5) leads to Definition (5.6) on the following page.

Figure 5.23



**Definition 5.6****Figure 5.24****Volume  $V$  of a Circular Disk 5.7**

$$V = \pi(\text{radius})^2 \cdot (\text{thickness})$$

Since we may revolve a region about the  $x$ -axis, the  $y$ -axis, or some other line, *it is not advisable to merely memorize the formulas in (5.5) and (5.6)*. It is better to remember the following general rule for finding the volume of a circular disk (see Figure 5.24).

**Guidelines for Finding the Volume  
of a Solid of Revolution  
Using Disks 5.8**

- 1 Sketch the region  $R$  to be revolved, and label the boundaries. Show a typical vertical rectangle of width  $dx$  or a horizontal rectangle of width  $dy$ .
- 2 Sketch the solid generated by  $R$  and the disk generated by the rectangle in guideline (1).
- 3 Express the radius of the disk in terms of  $x$  or  $y$ , depending on whether its thickness is  $dx$  or  $dy$ .
- 4 Use (5.7) to find a formula for the volume of the disk.
- 5 Apply the limit of sums operator  $\int_a^b$  or  $\int_c^d$  to the expression in guideline (4) and evaluate the integral.

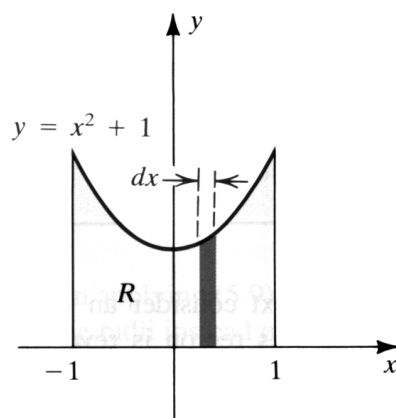
**EXAMPLE ■ I** The region bounded by the  $x$ -axis, the graph of the equation  $y = x^2 + 1$ , and the lines  $x = -1$  and  $x = 1$  is revolved about the  $x$ -axis. Find the volume of the resulting solid.

**SOLUTION** As specified in guideline (1), we sketch the region and show a vertical rectangle of width  $dx$  (see Figure 5.25a). Following guideline (2), we sketch the solid generated by  $R$  and the disk generated by the rectangle (see Figure 5.25b). As specified in guidelines (3) and (4), we note the following:

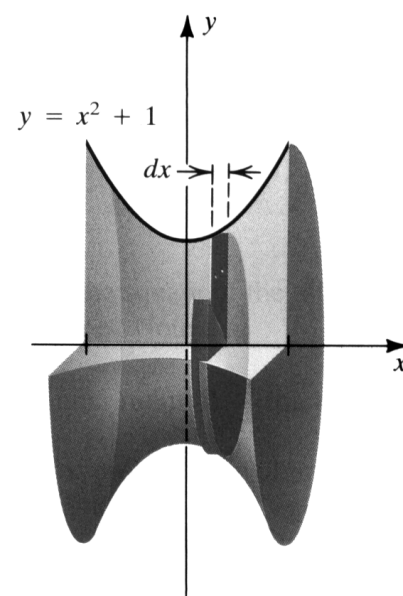
$$\begin{aligned} \text{thickness of disk: } & dx \\ \text{radius of disk: } & x^2 + 1 \\ \text{volume of disk: } & \pi(x^2 + 1)^2 dx \end{aligned}$$

Figure 5.25

(a)



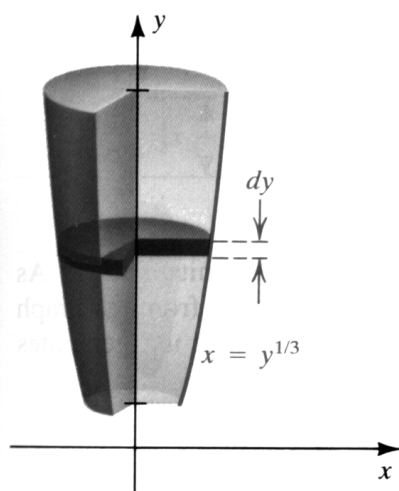
(b)



We could next apply guideline (5) with  $a = -1$  and  $b = 1$ , finding the volume  $V$  by regarding  $\int_{-1}^1$  as an operator that takes a limit of sums of volumes of disks. Another method is to use the symmetry of the region with respect to the  $y$ -axis and find  $V$  by applying  $\int_0^1$  to  $\pi(x^2 + 1)^2 dx$  and doubling the result. Thus,

$$\begin{aligned}
 V &= \int_{-1}^1 \pi(x^2 + 1)^2 dx \\
 &= 2 \int_0^1 \pi(x^4 + 2x^2 + 1) dx \\
 &= 2\pi \left[ \frac{x^5}{5} + 2 \left( \frac{x^3}{3} \right) + x \right]_0^1 \\
 &= 2\pi \left( \frac{1}{5} + \frac{2}{3} + 1 \right) = \frac{56}{15}\pi \approx 11.7.
 \end{aligned}$$

Figure 5.26



**EXAMPLE ■ 2** The region bounded by the  $y$ -axis and the graphs of  $y = x^3$ ,  $y = 1$ , and  $y = 8$  is revolved about the  $y$ -axis. Find the volume of the resulting solid.

**SOLUTION** The region and the solid are sketched in Figure 5.26, together with a disk generated by a typical horizontal rectangle. Since we plan to integrate with respect to  $y$ , we solve the equation  $y = x^3$  for  $x$  in terms of  $y$ , obtaining  $x = y^{1/3}$ . We note the following facts (see guidelines 3 and 4):

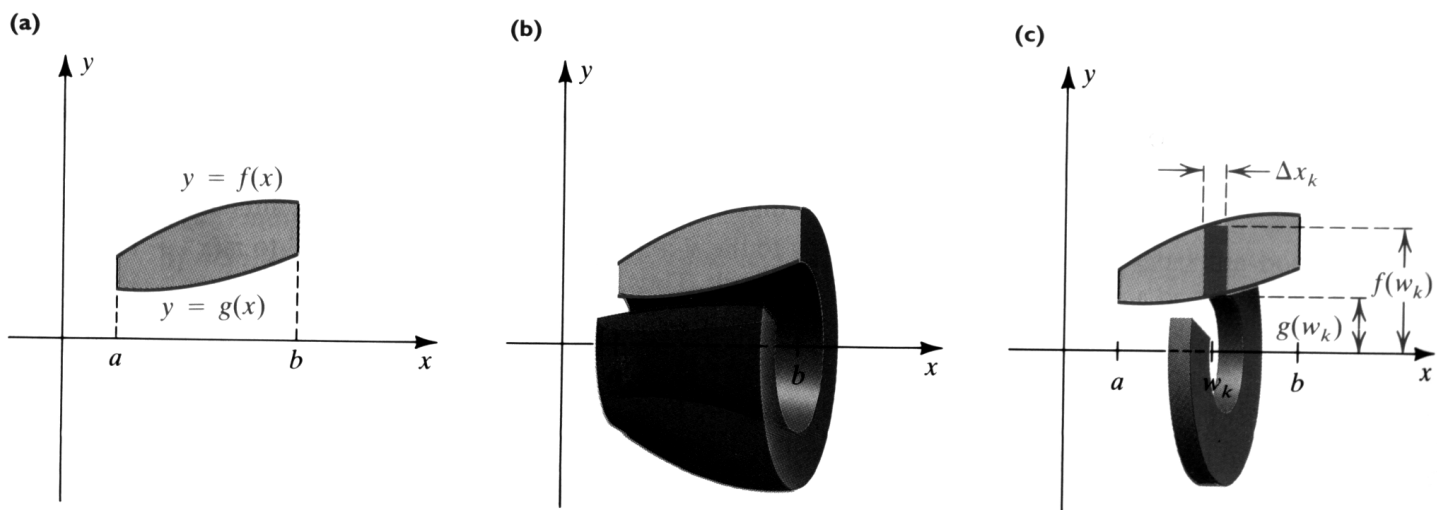
$$\begin{aligned}
 \text{thickness of disk:} & \quad dy \\
 \text{radius of disk:} & \quad y^{1/3} \\
 \text{volume of disk:} & \quad \pi(y^{1/3})^2 dy
 \end{aligned}$$

Finally, we apply guideline (5), with  $c = 1$  and  $d = 8$ , regarding  $\int_1^8$  as an operator that takes a limit of sums of disks:

$$\begin{aligned} V &= \int_1^8 \pi (y^{1/3})^2 dy = \pi \int_1^8 y^{2/3} dy = \pi \left[ \frac{y^{5/3}}{5/3} \right]_1^8 \\ &= \frac{3}{5} \pi \left[ y^{5/3} \right]_1^8 = \frac{3}{5} \pi [32 - 1] = \frac{93}{5} \pi \approx 58.4 \end{aligned}$$

Let us next consider an  $R_x$  region of the type illustrated in Figure 5.27(a). If this region is revolved about the  $x$ -axis, we obtain the solid illustrated in Figure 5.27(b). Note that if  $g(x) > 0$  for every  $x$  in  $[a, b]$ , there is a hole through the solid.

Figure 5.27



The volume  $V$  of the solid may be found by subtracting the volume of the solid generated by the smaller region from the volume of the solid generated by the larger region. Using Definition (5.5) gives us

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx - \int_a^b \pi [g(x)]^2 dx \\ &= \int_a^b \pi \{ [f(x)]^2 - [g(x)]^2 \} dx. \end{aligned}$$

The last integral has an interesting interpretation as a limit of sums. As illustrated in Figure 5.27(c), a vertical rectangle extending from the graph of  $g$  to the graph of  $f$ , through the points with  $x$ -coordinate  $w_k$ , generates a washer-shaped solid whose volume is

$$\pi [f(w_k)]^2 \Delta x_k - \pi [g(w_k)]^2 \Delta x_k = \pi \{ [f(w_k)]^2 - [g(w_k)]^2 \} \Delta x_k.$$

Summing the volumes of all such washers and taking the limit gives us the desired definite integral. When working problems of this type, it is convenient to use the following general rule.

### Volume $V$ of a Washer 5.9

$$V = \pi[(\text{outer radius})^2 - (\text{inner radius})^2] \cdot (\text{thickness})$$

In applying (5.9), a common error is to use the square of the difference of the radii instead of the difference of the squares. Note that

$$\text{volume of a washer} \neq \pi[(\text{outer radius}) - (\text{inner radius})]^2 \cdot (\text{thickness}).$$

Guidelines similar to (5.8) can be stated for problems involving washers. The principal differences are that in guideline (3), we find expressions for the outer radius and inner radius of a typical washer, and in guideline (4), we use (5.9) to find a formula for the volume of the washer.

**EXAMPLE ■ 3** The region bounded by the graphs of the equations  $x^2 = y - 2$  and  $2y - x - 2 = 0$  and the vertical lines  $x = 0$  and  $x = 1$  is revolved about the  $x$ -axis. Find the volume of the resulting solid.

**SOLUTION** The region and a typical vertical rectangle are sketched in Figure 5.28(a). Since we wish to integrate with respect to  $x$ , we solve the first two equations for  $y$  in terms of  $x$ , obtaining  $y = x^2 + 2$  and  $y = \frac{1}{2}x + 1$ . The solid and a washer generated by the rectangle are illustrated in Figure 5.28(b). Using (5.9) we obtain the following:

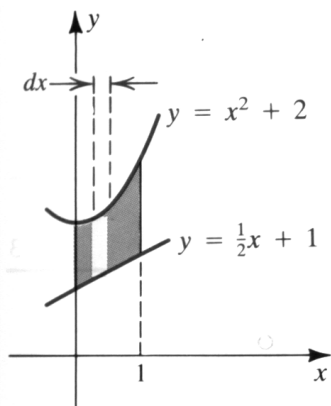
$$\begin{aligned} \text{thickness of washer: } & dx \\ \text{outer radius: } & x^2 + 2 \\ \text{inner radius: } & \frac{1}{2}x + 1 \\ \text{volume: } & \pi[(x^2 + 2)^2 - (\frac{1}{2}x + 1)^2] dx \end{aligned}$$

We take a limit of sums of volumes of washers by applying  $\int_0^1$ :

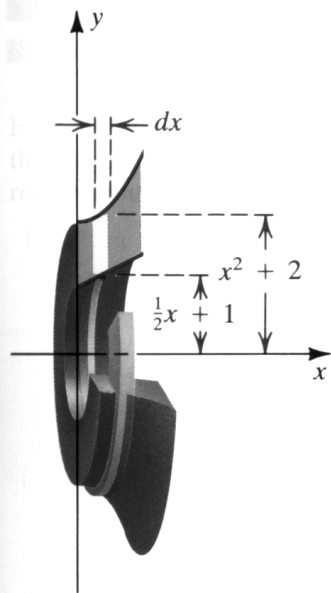
$$\begin{aligned} V &= \int_0^1 \pi[(x^2 + 2)^2 - (\frac{1}{2}x + 1)^2] dx \\ &= \pi \int_0^1 (x^4 + \frac{15}{4}x^2 - x + 3) dx \\ &= \pi \left[ \frac{x^5}{5} + \frac{15}{4} \left( \frac{x^3}{3} \right) - \frac{x^2}{2} + 3x \right]_0^1 = \frac{79\pi}{20} \approx 12.4 \end{aligned}$$

Figure 5.28

(a)



(b)

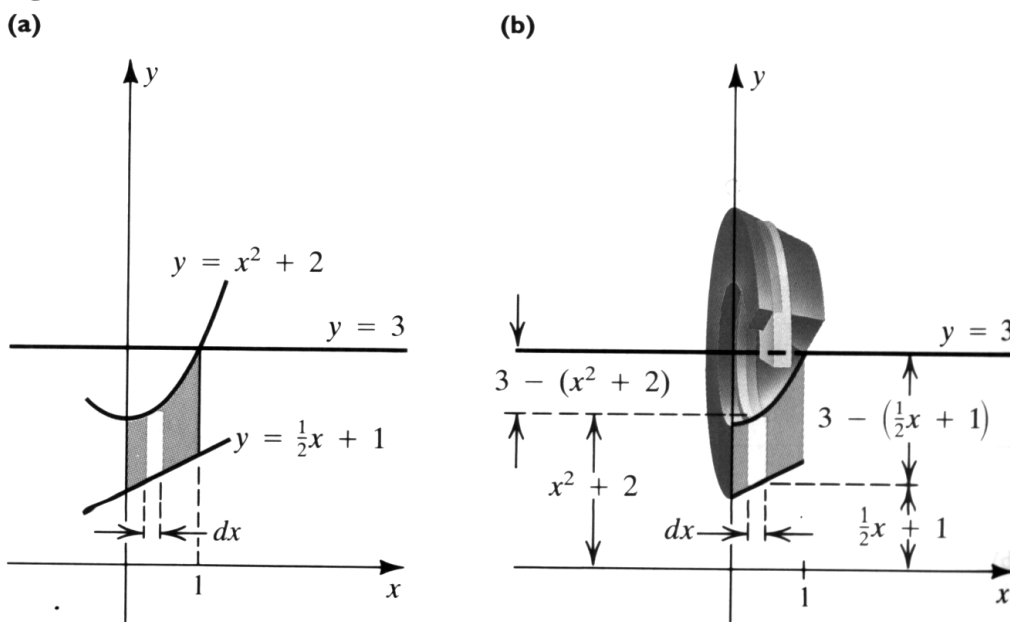


**EXAMPLE ■ 4** Find the volume of the solid generated by revolving the region described in Example 3 about the line  $y = 3$ .

**SOLUTION** The region and a typical vertical rectangle are re-sketched in Figure 5.29(a), together with the axis of revolution  $y = 3$ . The solid and a washer generated by the rectangle are illustrated in Figure 5.29(b). We note the following:

$$\begin{aligned} \text{thickness of washer: } & dx \\ \text{outer radius: } & 3 - \left(\frac{1}{2}x + 1\right) = 2 - \frac{1}{2}x \\ \text{inner radius: } & 3 - (x^2 + 2) = 1 - x^2 \\ \text{volume: } & \pi[(2 - \frac{1}{2}x)^2 - (1 - x^2)^2] dx \end{aligned}$$

**Figure 5.29**

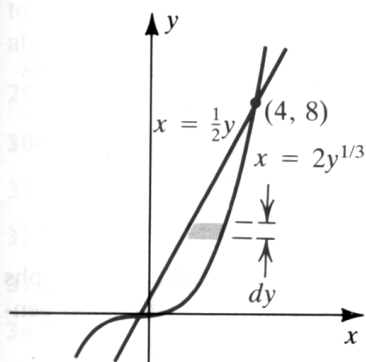


Applying the limit of sums operator  $\int_0^1$  gives us the volume:

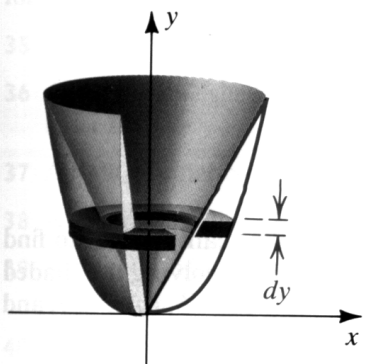
$$\begin{aligned} V &= \int_0^1 \pi[(2 - \frac{1}{2}x)^2 - (1 - x^2)^2] dx \\ &= \pi \int_0^1 (3 - 2x + \frac{9}{4}x^2 - x^4) dx \\ &= \pi \left[ 3x - x^2 + \frac{9}{4} \left( \frac{x^3}{3} \right) - \frac{x^5}{5} \right]_0^1 \\ &= \frac{51\pi}{20} \approx 8.01 \end{aligned}$$

Figure 5.30

(a)



(b)



**EXAMPLE ■ 5** The region in the first quadrant bounded by the graphs of  $y = \frac{1}{8}x^3$  and  $y = 2x$  is revolved about the  $y$ -axis. Find the volume of the resulting solid.

**SOLUTION** The region and a typical horizontal rectangle are shown in Figure 5.30(a). We wish to integrate with respect to  $y$ , so we solve the given equations for  $x$  in terms of  $y$ , obtaining

$$x = \frac{1}{2}y \quad \text{and} \quad x = 2y^{1/3}.$$

Figure 5.30(b) illustrates the volume generated by the region and the washer generated by the rectangle. We note the following:

thickness of washer:  $dy$

outer radius:  $2y^{1/3}$

inner radius:  $\frac{1}{2}y$

$$\text{volume: } \pi[(2y^{1/3})^2 - (\frac{1}{2}y)^2] dy = \pi(4y^{2/3} - \frac{1}{4}y^2) dy$$

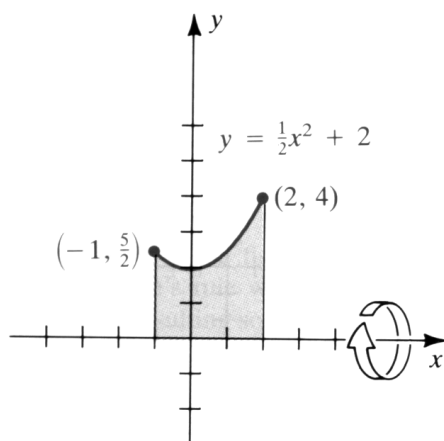
Applying the limit of sums operator  $\int_0^8$  gives us the volume:

$$\begin{aligned} V &= \int_0^8 \pi(4y^{2/3} - \frac{1}{4}y^2) dy \\ &= \pi \left[ \frac{12}{5}y^{5/3} - \frac{1}{12}y^3 \right]_0^8 = \frac{512}{15}\pi \approx 107.2 \end{aligned}$$

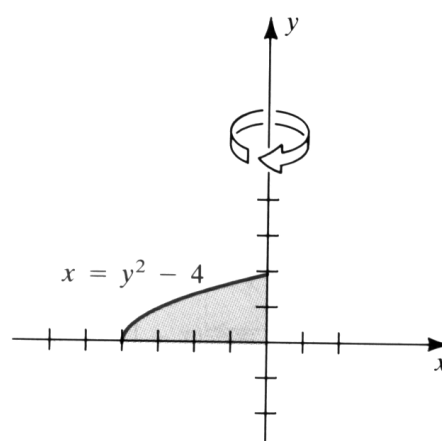
## EXERCISES 5.2

Exer. 1–4: Set up an integral that can be used to find the volume of the solid obtained by revolving the shaded region about the indicated axis.

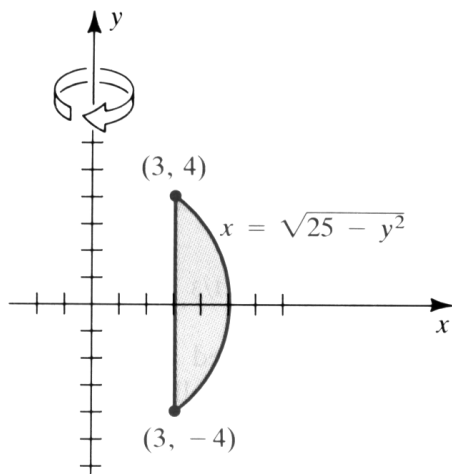
1



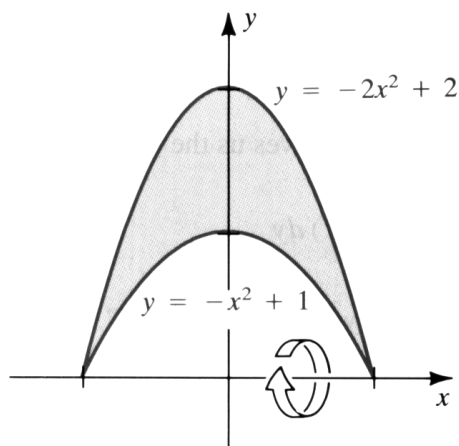
2



3



4



**Exer. 5–24:** Sketch the region  $R$  bounded by the graphs of the equations, and find the volume of the solid generated if  $R$  is revolved about the indicated axis.

5  $y = 1/x$ ,  $x = 1$ ,  $x = 3$ ,  $y = 0$ ;  $x$ -axis

6  $y = \sqrt{x}$ ,  $x = 4$ ,  $y = 0$ ;  $x$ -axis

7  $y = x^2 - 4x$ ,  $y = 0$ ;  $x$ -axis

8  $y = x^3$ ,  $x = -2$ ,  $y = 0$ ;  $x$ -axis

9  $y = x^2$ ,  $y = 2$ ;  $y$ -axis

10  $y = 1/x$ ,  $y = 1$ ,  $y = 3$ ,  $x = 0$ ;  $y$ -axis

11  $x = 4y - y^2$ ,  $x = 0$ ;  $y$ -axis

12  $y = x$ ,  $y = 3$ ,  $x = 0$ ;  $y$ -axis

13  $y = x^2$ ,  $y = 4 - x^2$ ;  $x$ -axis

14  $x = y^3$ ,  $x^2 + y = 0$ ;  $x$ -axis

15  $y = x$ ,  $x + y = 4$ ,  $x = 0$ ;  $x$ -axis

16  $y = (x - 1)^2 + 1$ ,  $y = -(x - 1)^2 + 3$ ;  $x$ -axis

17  $y^2 = x$ ,  $2y = x$ ;  $y$ -axis

18  $y = 2x$ ,  $y = 4x^2$ ;  $y$ -axis

19  $x = y^2$ ,  $x - y = 2$ ;  $y$ -axis

20  $x + y = 1$ ,  $x - y = -1$ ,  $x = 2$ ;  $y$ -axis

21  $y = \sin 2x$ ,  $x = 0$ ,  $x = \pi$ ,  $y = 0$ ;  $x$ -axis  
(Hint: Use a half-angle formula.)

22  $y = 1 + \cos 3x$ ,  $x = 0$ ,  $x = 2\pi$ ,  $y = 0$ ;  $x$ -axis  
(Hint: Use a half-angle formula.)

23  $y = \sin x$ ,  $y = \cos x$ ,  $x = 0$ ,  $x = \pi/4$ ;  $x$ -axis  
(Hint: Use a double angle formula.)

24  $y = \sec x$ ,  $y = \sin x$ ,  $x = 0$ ,  $x = \pi/4$ ;  $x$ -axis

**Exer. 25–26:** Sketch the region  $R$  bounded by the graphs of the equations, and find the volume of the solid generated if  $R$  is revolved about the given line.

25  $y = x^2$ ,  $y = 4$

(a)  $y = 4$  (b)  $y = 5$

(c)  $x = 2$  (d)  $x = 3$

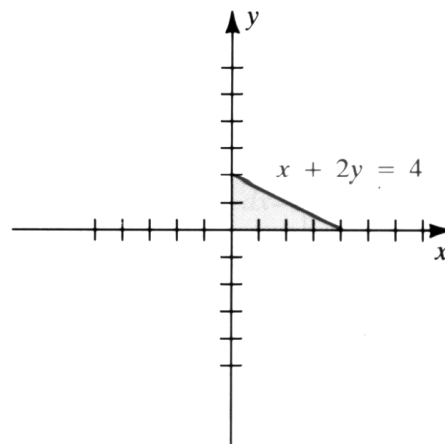
26  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 4$

(a)  $x = 4$  (b)  $x = 6$

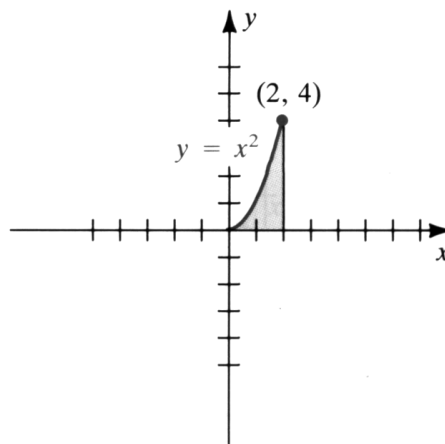
(c)  $y = 2$  (d)  $y = 4$

**Exer. 27–28:** Set up an integral that can be used to find the volume of the solid generated by revolving the shaded region about the line (a)  $y = -2$ , (b)  $y = 5$ , (c)  $x = 7$ , and (d)  $x = -4$ .

27



28



**Exer. 29–34:** Sketch the region  $R$  bounded by the graphs of the equations, and set up integrals that can be used to find the volume of the solid generated if  $R$  is revolved about the given line.

29  $y = x^3$ ,  $y = 4x$ ;  $y = 8$

30  $y = x^3$ ,  $y = 4x$ ;  $x = 4$

31  $x + y = 3$ ,  $y + x^2 = 3$ ;  $x = 2$

32  $y = 1 - x^2$ ,  $x - y = 1$ ;  $y = 3$

33  $x^2 + y^2 = 1$ ;  $x = 5$

34  $y = x^{2/3}$ ,  $y = x^2$ ;  $y = -1$

**Exer. 35–40:** Use a definite integral to derive a formula for the volume of the indicated solid.

35 A right circular cylinder of altitude  $h$  and radius  $r$

36 A cylindrical shell of altitude  $h$ , outer radius  $R$ , and inner radius  $r$

37 A right circular cone of altitude  $h$  and base radius  $r$

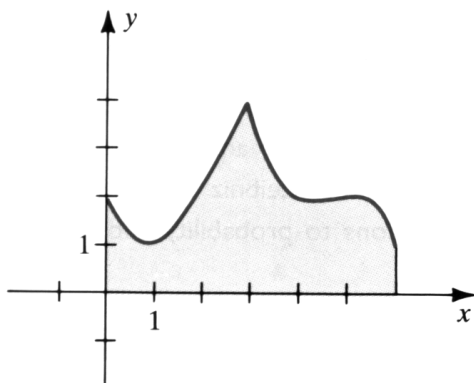
38 A sphere of radius  $r$

39 A frustum of a right circular cone of altitude  $h$ , lower base radius  $R$ , and upper base radius  $r$

40 A spherical segment of altitude  $h$  in a sphere of radius  $r$

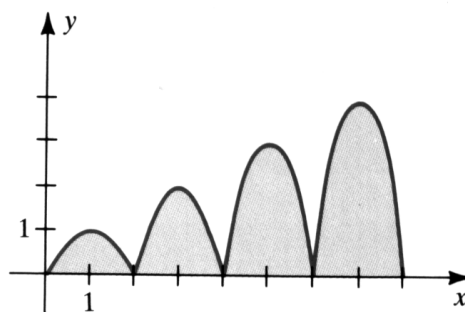
41 If the region shown in the figure is revolved about the  $x$ -axis, use the trapezoidal rule with  $n = 6$  to approximate the volume of the resulting solid.

#### Exercise 41



42 If the region shown in the figure is revolved about the  $x$ -axis, use Simpson's rule with  $n = 4$  to approximate the volume of the resulting solid.

#### Exercise 42



**Exer. 43–44:** Graph  $f$  and  $g$  on the same coordinate axes for  $0 \leq x \leq \pi$ . (a) Estimate the  $x$ -coordinates of the points of intersection of the graphs. (b) If the region bounded by the graphs of  $f$  and  $g$  is revolved about the  $x$ -axis, use Simpson's rule with  $n = 2$  to approximate the volume of the resulting solid.

43  $f(x) = \frac{\sin x}{1+x}$ ;  $g(x) = 0.3$

44  $f(x) = \sqrt[4]{|\sin x|}$ ;  $g(x) = 0.2x + 0.7$

45 Find the volume of the solid obtained by revolving the region bounded by the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  about the  $x$ -axis.

46 Work Exercise 45 with the region revolved about the  $y$ -axis.

47 A *paraboloid of revolution* is formed by revolving a parabola about its axis. Paraboloids are the basic shape for a wide variety of collectors and reflectors. Shown in the figure is a (finite) paraboloid of altitude  $h$  and radius of base  $r$ .

(a) The *focal length* of the paraboloid is the distance  $p$  between the vertex and the focus of the parabola. Express  $p$  in terms of  $r$  and  $h$ .

(b) Find the volume of the paraboloid.

#### Exercise 47

