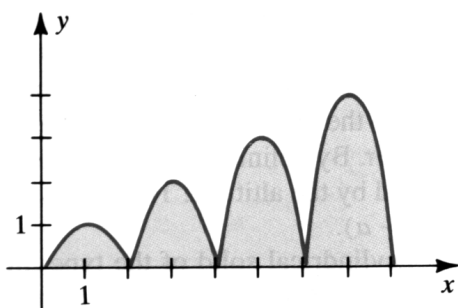


### Exercise 32



**c** 33 Graph  $f(x) = -x^4 + 2.21x^3 - 3.21x^2 + 4.42x - 2$ .

- Estimate the  $x$ -intercepts of the graph.
- If the region bounded by the graph of  $f$  and the  $x$ -axis is revolved about the  $y$ -axis, set up an integral that can be used to approximate the volume of the resulting solid.

**c** 34 Graph, on the same coordinate axes,  $f(x) = \csc x$  and  $g(x) = x + 1$  for  $0 < x < \pi$ .

- Use Newton's method to approximate, to four decimal places, the  $x$ -coordinates of the points of intersection of the graphs.
- If the region bounded by the graphs is revolved

about the  $y$ -axis, use the trapezoidal rule, with  $n = 6$ , to approximate the volume of the resulting solid.

35 Let  $R$  be the region bounded by the parabola  $x^2 = 4y$  and the line  $l$  through the focus that is perpendicular to the axis of the parabola.

- Find the area of  $R$ .
- If  $R$  is revolved about the  $y$ -axis, find the volume of the resulting solid.
- If  $R$  is revolved about the  $x$ -axis, find the volume of the resulting solid.

36 Work (a)–(c) of Exercise 35 if  $R$  is the region bounded by the graphs of  $y^2 = 2x - 6$  and  $x = 5$ .

**Exer. 37–38:** Let  $R$  be the region bounded by the hyperbola with equation  $b^2x^2 - a^2y^2 = a^2b^2$  and a vertical line through a focus.

37 Show that the area of the region  $R$  is given by

$$\frac{2b}{a} \int_a^c \sqrt{x^2 - a^2} dx, \text{ where } c = \sqrt{a^2 + b^2}.$$

38 Find the volume of the solid obtained by revolving  $R$  about the  $y$ -axis.

## 5.4

## VOLUMES BY CROSS SECTIONS



If a plane intersects a solid, then the region common to the plane and the solid is a **cross section** of the solid. In Section 5.2, we used circular and washer-shaped cross sections to find volumes of solids of revolution. In this section, we shall study solids that have the following property (see Figure 5.36): For every  $x$  in  $[a, b]$ , the plane perpendicular to the  $x$ -axis at  $x$  intersects the solid in a cross section whose area is  $A(x)$ , where  $A$  is a continuous function on  $[a, b]$ .

Figure 5.36

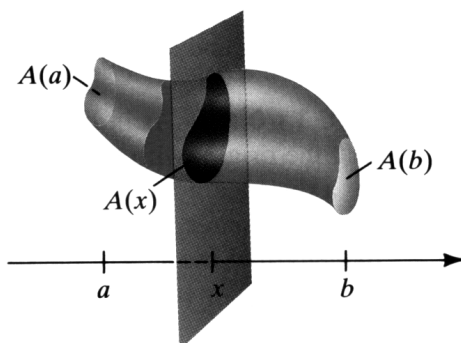


Figure 5.37

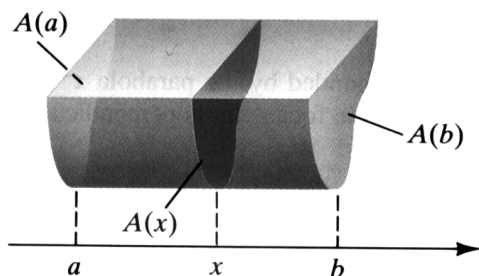
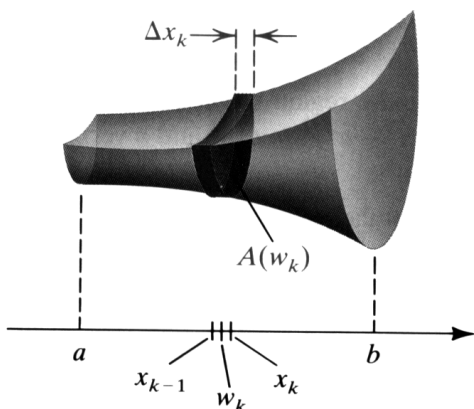


Figure 5.38



The solid is called a **cylinder** if, as illustrated in Figure 5.37 a line parallel to the  $x$ -axis that traces the boundary of the cross section corresponding to  $a$  also traces the boundary of the cross section corresponding to every  $x$  in  $[a, b]$ . The cross sections determined by the planes through  $x = a$  and  $x = b$  are the **bases** of the cylinder. The distance between the bases is the **altitude** of the cylinder. By definition, the volume of the cylinder is the area of a base multiplied by the altitude. Thus, the volume of the solid in Figure 5.37 is  $A(a) \cdot (b - a)$ .

To find the volume of a noncylindrical solid of the type illustrated in Figure 5.38, we begin with a partition  $P$  of  $[a, b]$ . Planes perpendicular to the  $x$ -axis at each  $x_k$  in the partition slice the solid into smaller pieces. If we choose any number  $w_k$  in  $[x_{k-1}, x_k]$ , the volume of a typical slice can be approximated by the volume  $A(w_k)\Delta x_k$  of the red cylinder shown in Figure 5.38. If  $V$  is the volume of the solid and if the norm  $\|P\|$  is small, then

$$V \approx \sum_k A(w_k)\Delta x_k.$$

Since this approximation improves as  $\|P\|$  gets smaller, we define the volume of the solid by

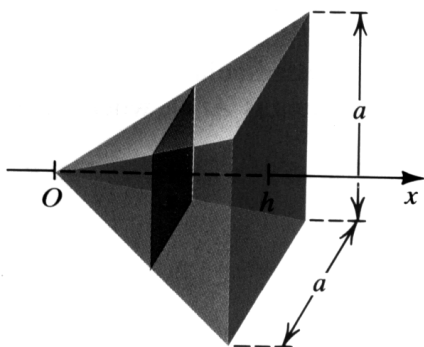
$$V = \lim_{\|P\| \rightarrow 0} \sum_k A(w_k)\Delta x_k = \int_a^b A(x) dx,$$

where the last equality follows from the definition of the definite integral. We may summarize our discussion as follows.

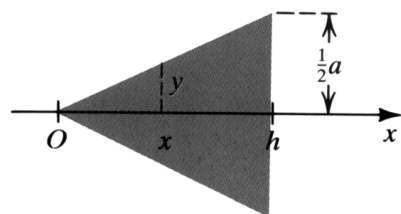
### Volumes by Cross Sections 5.13

Figure 5.39

(a)



(b)



Let  $S$  be a solid bounded by planes that are perpendicular to the  $x$ -axis at  $a$  and  $b$ . If, for every  $x$  in  $[a, b]$ , the cross-sectional area of  $S$  is given by  $A(x)$ , where  $A$  is continuous on  $[a, b]$ , then the volume of  $S$  is

$$V = \int_a^b A(x) dx.$$

An analogous result can be stated for a  $y$ -interval  $[c, d]$  and a cross-sectional area  $A(y)$ .

**EXAMPLE ■ I** Find the volume of a right pyramid with a square base of side  $a$  and altitude  $h$ .

**SOLUTION** As in Figure 5.39(a), let us take the vertex of the pyramid at the origin, with the  $x$ -axis passing through the center of the square base, a distance  $h$  from  $O$ . Cross sections by planes perpendicular to the  $x$ -axis are squares. Figure 5.39(b) is a side view of the pyramid. Since  $2y$  is the length of the side of the square cross section corresponding to  $x$ , the cross-sectional area  $A(x)$  is

$$A(x) = (2y)^2 = 4y^2.$$

Using similar triangles in Figure 5.39(b), we have

$$\frac{y}{x} = \frac{\frac{1}{2}a}{h}, \quad \text{or} \quad y = \frac{ax}{2h}.$$

Hence,

$$A(x) = 4y^2 = \frac{4a^2x^2}{4h^2} = \frac{a^2}{h^2}x^2.$$

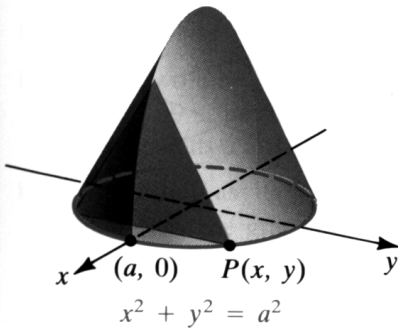
Applying (5.13) yields

$$\begin{aligned} V &= \int_0^h A(x) dx = \int_0^h \left( \frac{a^2}{h^2} \right) x^2 dx \\ &= \left( \frac{a^2}{h^2} \right) \left[ \frac{x^3}{3} \right]_0^h = \frac{a^2}{h^2} \frac{h^3}{3} = \frac{1}{3}a^2h. \end{aligned}$$

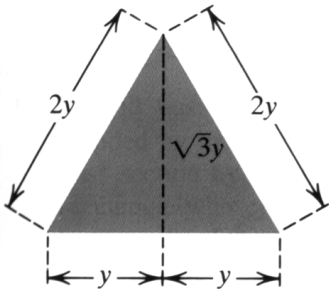
**EXAMPLE ■ 2** A solid has, as its base, the circular region in the  $xy$ -plane bounded by the graph of  $x^2 + y^2 = a^2$  with  $a > 0$ . Find the volume of the solid if every cross section by a plane perpendicular to the  $x$ -axis is an equilateral triangle with one side in the base.

Figure 5.40

(a)



(b)



**SOLUTION** A triangular cross section by a plane  $x$  units from the origin is illustrated in Figure 5.40(a). If the point  $P(x, y)$  is on the circle and  $y > 0$ , then the lengths of the sides of this equilateral triangle are  $2y$ . Referring to Figure 5.40(b), we see, by the Pythagorean theorem, that the altitude of the triangle is

$$\sqrt{(2y)^2 - y^2} = \sqrt{3y^2} = \sqrt{3}y.$$

Hence, the area  $A(x)$  of the cross section is

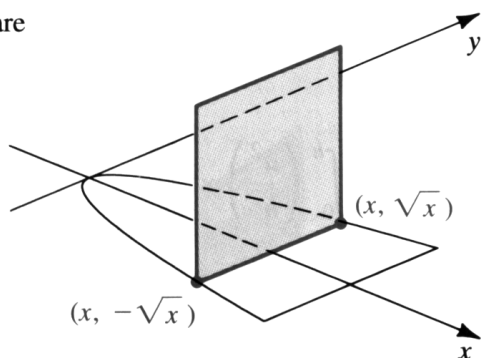
$$A(x) = \frac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2 = \sqrt{3}(a^2 - x^2).$$

Applying (5.13) gives us

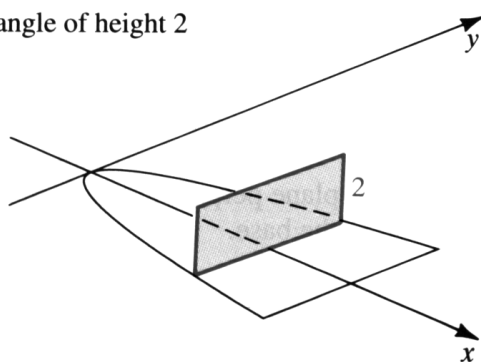
$$\begin{aligned} V &= \int_{-a}^a A(x) dx = \int_{-a}^a \sqrt{3}(a^2 - x^2) dx \\ &= \sqrt{3} \left[ a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{4\sqrt{3}}{3}a^3. \end{aligned}$$

Exer. 1–8: Let  $R$  be the region bounded by the graphs of  $x = y^2$  and  $x = 9$ . Find the volume of the solid that has  $R$  as its base if every cross section by a plane perpendicular to the  $x$ -axis has the given shape.

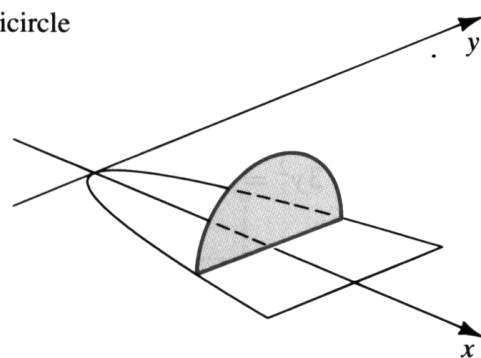
1 A square



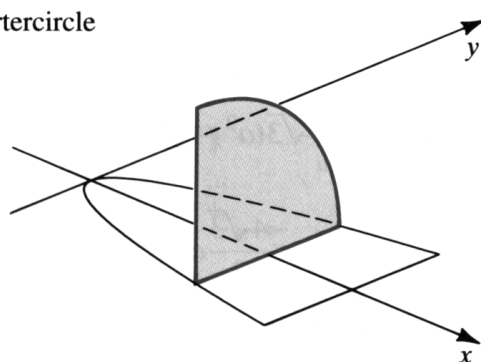
2 A rectangle of height 2



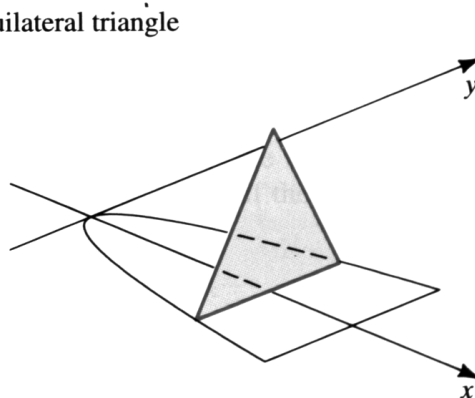
3 A semicircle



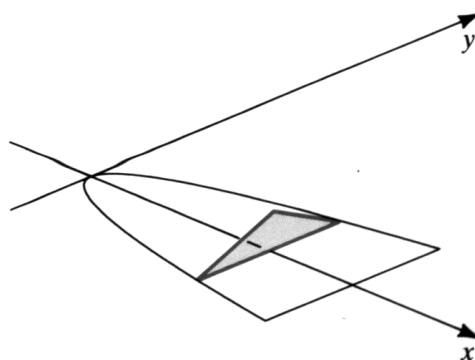
4 A quartercircle



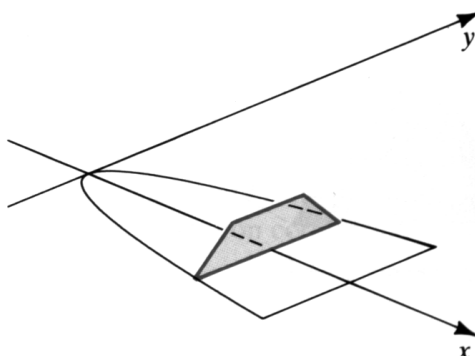
5 An equilateral triangle



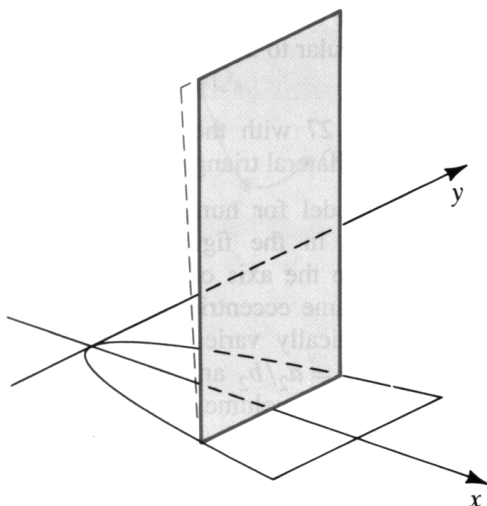
6 A triangle with height equal to  $\frac{1}{4}$  the length of the base



7 A trapezoid with lower base in the  $xy$ -plane, upper base equal to  $\frac{1}{2}$  the length of the lower base, and height equal to  $\frac{1}{4}$  the length of the lower base

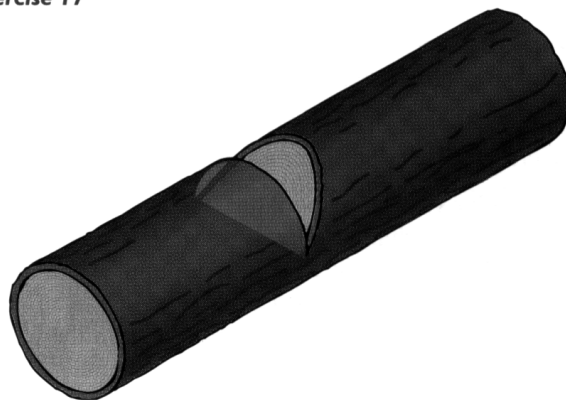


- 8 A parallelogram with base in the  $xy$ -plane and height equal to twice the length of the base



angle of  $45^\circ$ , both cuts intersecting at the center of the log (see figure). Find the volume of the wedge.

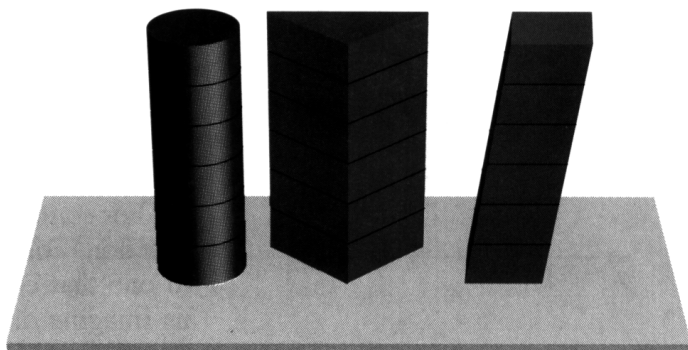
### Exercise 17



- 9 A solid has as its base the circular region in the  $xy$ -plane bounded by the graph of  $x^2 + y^2 = a^2$  with  $a > 0$ . Find the volume of the solid if every cross section by a plane perpendicular to the  $x$ -axis is a square.
- 10 Work Exercise 9 if every cross section is an isosceles triangle with base on the  $xy$ -plane and altitude equal to the length of the base.
- 11 A solid has as its base the region in the  $xy$ -plane bounded by the graphs of  $y = 4$  and  $y = x^2$ . Find the volume of the solid if every cross section by a plane perpendicular to the  $x$ -axis is an isosceles right triangle with hypotenuse on the  $xy$ -plane.
- 12 Work Exercise 11 if every cross section is a square.
- 13 Find the volume of a pyramid of the type illustrated in Figure 5.39 if the altitude is  $h$  and the base is a rectangle of dimensions  $a$  and  $2a$ .
- 14 A solid has as its base the region in the  $xy$ -plane bounded by the graphs of  $y = x$  and  $y^2 = x$ . Find the volume of the solid if every cross section by a plane perpendicular to the  $x$ -axis is a semicircle with diameter in the  $xy$ -plane.
- 15 A solid has as its base the region in the  $xy$ -plane bounded by the graphs of  $y^2 = 4x$  and  $x = 4$ . If every cross section by a plane perpendicular to the  $y$ -axis is a semicircle, find the volume of the solid.
- 16 A solid has as its base the region in the  $xy$ -plane bounded by the graphs of  $x^2 = 16y$  and  $y = 2$ . Every cross section by a plane perpendicular to the  $y$ -axis is a rectangle whose height is twice that of the side in the  $xy$ -plane. Find the volume of the solid.
- 17 A log having the shape of a right circular cylinder of radius  $a$  is lying on its side. A wedge is removed from the log by making a vertical cut and another cut at an

- 18 The axes of two right circular cylinders of radius  $a$  intersect at right angles. Find the volume of the solid bounded by the cylinders.
- 19 The base of a solid is the circular region in the  $xy$ -plane bounded by the graph of  $x^2 + y^2 = a^2$  with  $a > 0$ . Find the volume of the solid if every cross section by a plane perpendicular to the  $x$ -axis is an isosceles triangle of constant altitude  $h$ . (Hint: Interpret  $\int_{-a}^a \sqrt{a^2 - x^2} dx$  as an area.)
- 20 Cross sections of a horn-shaped solid by planes perpendicular to its axis are circles. If a cross section that is  $s$  inches from the smaller end of the solid has diameter  $6 + \frac{1}{36}s^2$  inches and if the length of the solid is 2 ft, find its volume.
- 21 A tetrahedron has three mutually perpendicular faces and three mutually perpendicular edges of lengths 2, 3, and 4 cm, respectively. Find its volume.
- 22 *Cavalieri's theorem* states that if two solids have equal altitudes and if all cross sections by planes parallel to their bases and at the same distances from their bases have equal areas, then the solids have the same volume (see figure). Prove Cavalieri's theorem.

### Exercise 22

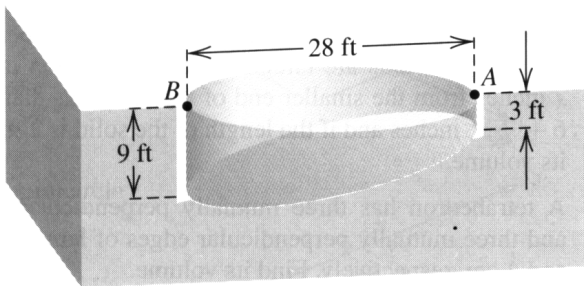


- 23** The base of a solid is an isosceles right triangle whose equal sides have length  $a$ . Find the volume if cross sections that are perpendicular to the base and to one of the equal sides are semicircular.
- 24** Work Exercise 23 if the cross sections are regular hexagons with one side in the base.
- 25** Show that the disk and washer methods discussed in Section 5.2 are special cases of (5.13).
- c 26** A circular swimming pool has diameter 28 ft. The depth of the water changes slowly from 3 ft at a point  $A$  on one side of the pool to 9 ft at a point  $B$  diametrically opposite  $A$  (see figure). Depth readings  $h(x)$  (in feet) taken along the diameter  $AB$  are given in the following table, where  $x$  is the distance (in feet) from  $A$ .

$x$	0	4	8	12	16	20	24	28
$h(x)$	3	3.5	4	5	6.5	8	8.5	9

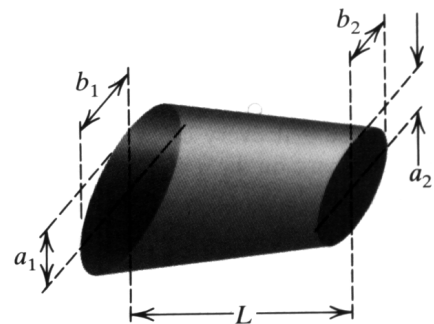
Use the trapezoidal rule, with  $n = 7$ , to estimate the volume of water in the pool. Approximate the number of gallons of water contained in the pool ( $1 \text{ gal} \approx 0.134 \text{ ft}^3$ ).

**Exercise 26**



- 27** The base of a solid is a region bounded by an ellipse with major and minor axes of lengths 16 and 9, respectively. Find the volume of the solid if every cross section by a plane perpendicular to the major axis has the shape of a square.
- 28** Work Exercise 27 with the cross section having the shape of an equilateral triangle.
- 29** A common model for human limbs is the *elliptical frustum* shown in the figure, where cross sections perpendicular to the axis of the frustum are elliptical and have the same eccentricity. For human limbs, the eccentricity typically varies from 0.6 to values near 1. If  $k = a_1/b_1 = a_2/b_2$  and if  $L$  is the length of the limb, show that the volume  $V$  is given by the equation  $V = (\frac{1}{3}\pi L/k)(a_1^2 + a_1a_2 + a_2^2)$ . (Hint: Use Exercise 39 in Section 5.1.)

**Exercise 29**



- 30** The base of a right elliptic cone has major and minor axes of lengths  $2a$  and  $2b$ , respectively. Find the volume if the altitude of the cone is  $h$ . (Hint: Use Exercise 39 in Section 5.1.)