

4.1: 1, 4, 5, 8, 9, 13

4.2: 1, 2, 10, 22, 28, 30, 41-44

4.3: 1, 2, 4, 8, 22, 26, 30

4.1

① $f(x) = x^3 - x$ on $[0, 1]$

f is continuous & differentiable everywhere (it's a polynomial and $f(0) = 0 = f(1)$). Thus the hypotheses for Rolle's Theorem are satisfied.

$$f'(x) = 3x^2 - 1$$

To find an $x=c$ where $f'(c)=0$, we set

$$0 = 3c^2 - 1$$

$$\Rightarrow c^2 = 1/3 \Rightarrow c = \pm 1/\sqrt{3}$$

$c = -1/\sqrt{3}$ is coincidentally another place where $f'(c)=0$
 $0 < c = 1/\sqrt{3} < 1$ is the solution which verifies Rolle's Theorem.

④ $f(x) = x^{2/3} - 2x^{1/3} \left(= \left(\sqrt[3]{x}\right)^2 - 2\left(\sqrt[3]{x}\right) \right)$ on $[0, 8]$

f looks like a polynomial other than for the cube roots. But the cube root function is continuous everywhere. Since polynomials are continuous everywhere too, and since the composition of continuous functions is continuous, then f is continuous everywhere. Likewise, the cube root function is differentiable everywhere but at $x=0$ (it has a vertical tangent there), polynomials are differentiable everywhere, and the composition of differentiable functions is differentiable. Thus f is differentiable at least everywhere but at $x=0$. Thus the conditions for Rolle's Theorem are satisfied (f is continuous on

④ (continued) $[0, 8]$ & differentiable on $(0, 8)$ and $f(0) = 0 = f(8)$

$$f'(x) = \frac{2}{3} x^{-1/3} - \frac{2}{3} x^{-2/3} \quad (\text{except when } x=0)$$

$$= \frac{2}{3} x^{-1/3} (1 - x^{-1/3})$$

To find an $x=c$ where $f'(c)=0$, we set

$$0 = \frac{2}{3} c^{-1/3} (1 - c^{-1/3})$$

$$\Rightarrow \frac{2}{3} = 0 \quad (\text{never true})$$

$$\text{or } c^{-1/3} = 0 \quad (\text{never true})$$

$$\text{or } 1 - c^{-1/3} = 0 \quad (\text{true only when } c=1)$$

Thus $0 < c=1 < 8$ is the sol'n which verifies the conclusion of Rolle's theorem.

⑤ $f(x) = x^2$ on $[1, 2]$

f is continuous & differentiable everywhere so it satisfies

The hypotheses of the mean value theorem.

$$f(1)=1 \quad f(2)=4 \Rightarrow \frac{f(2)-f(1)}{2-1} = 3$$

To verify the MVT, we need to find an $x=c$ so that $f'(c)=3$ and $1 < c < 2$. Since $f'(x)=2x$, this is the same as finding $2c=3$ with $1 < c < 2$. We certainly have that with $1 < c = 3/2 < 2$.

⑥ Following the reasoning in ④, $f(x) = x^{2/3}$ is continuous everywhere and differentiable at least everywhere but at $x=0$. Thus f is certainly continuous on $[1, 8]$ & differentiable on $(1, 8)$ so the conditions of the MVT are satisfied. To verify the conclusion of the MVT, we must find an $x=c$ so that $f'(c) = \frac{f(8)-f(1)}{8-1}$ with $1 < c < 8$

⑧ (continued). In this exercise, that is tantamount to solving:

$$\frac{2}{3} c^{-1/3} = \frac{4-1}{8-1}$$

$$\Rightarrow c^{-1/3} = \frac{3}{2} \cdot \frac{3}{7}$$

$$\Rightarrow c = \frac{2764}{729}$$

Since $1 < \frac{2764}{729} < 8$, the conclusion of the MVT is verified.
 ≈ 4

⑨ Following ④, $f(x) = \sqrt{1-x^2} = g(h(x))$ where $g(x) = \sqrt{x}$ and $h(x) = 1-x^2$. Since g is continuous everywhere it is defined — the non-negative numbers: $[0, \infty)$, and since h is continuous everywhere, then f is continuous on $[-1, 1]$. Since g is diff. on $(0, \infty)$ & h is diff. everywhere, h is at least diff. on $(-1, 1)$. Thus the hypotheses of the MVT are satisfied for this problem (that is, f is cont. on $[0, 1]$ & diff. on $(0, 1)$). To verify the conclusion of the MVT, we need to find an $x=c$ such that $f'(c) = \frac{f(1)-f(0)}{1-0}$ and $0 < c < 1$.

In this case, we want

$$\frac{1}{2}(1-c^2)^{-1/2} \cdot (-2c) = \frac{0-1}{1-0} \quad \text{with } 0 < c < 1.$$

$$\Rightarrow \frac{-c}{\sqrt{1-c^2}} = -1 \Rightarrow -c = -\sqrt{1-c^2} \Rightarrow c^2 = 1-c^2$$

so long as $c \neq \pm 1$ (and we have introduced an extraneous sol'n)

$$\Rightarrow c = \pm 1/\sqrt{2}. \quad \text{Since } f'(1/\sqrt{2}) = 1, \quad \text{~~this~~ } x = -1/\sqrt{2} \text{ is extraneous.}$$

However $f'(1/\sqrt{2}) = -1$ and $0 < 1/\sqrt{2} < 1$ so the conclusion of the MVT is indeed true.

- ⑬ The question begins to read: "Does there exist a differentiable function $f \dots$ " This is generally taken to read as (the more specific): "Does there exist a function f which is differentiable everywhere...."

Suppose we have a function f which is differentiable everywhere, $f(0)=2$, and $f(2)=5$. Further suppose that $f'(x) \leq 1$ for all $x \in (0,2)$. Since f is differentiable everywhere, it is continuous everywhere. Thus f satisfies the hypotheses of the MVT for the interval $[0,2]$. The MVT then tells us that there ^{must be} an $x=c$ such that $f'(c) = \frac{f(2)-f(0)}{2-0}$ with $0 < c < 2$.

That is, there must be an $x=c$ such that $f'(c) = 3/2$ with $0 < c < 2$. This contradicts the ~~supposition~~ supposition that $f'(x) \leq 1$ for all x with $0 < x < 2$. Thus at least one of our suppositions must be untrue, and the answer to the posed question in the exercise is "no." The "why" is the above reasoning.

4.2

f increases when $f' > 0$; f decreases when $f' < 0$. f' can only change sign in a range of inputs if f' is 0 or f' is discontinuous. Thus finding the values for which f' is 0 or discontinuous will allow us to partition the range of inputs into subsets; in each of these subsets, f' will either

always be positive or f' will always be negative.

① $f(x) = x^3 - 3x + 2$

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

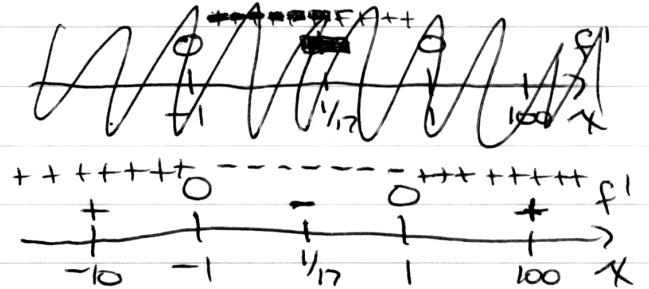
f' is continuous everywhere

f' is 0 for $x = \pm 1$ (solve $3x^2 - 3 = 0$)

$$f'(-10) = 297 > 0$$

$$f'(1/17) = -3 + \frac{3}{289} < 0$$

$$f'(100) = 29997 > 0$$



Thus f is inc on $(-\infty, -1)$ and on $(1, \infty)$

and f is dec on $(-1, 1)$

f is neither inc nor dec at ± 1 .

② $f(x) = x^3 - 3x^2 + 6$

$$f'(x) = 3x^2 - 6x = 3x(x-6)$$

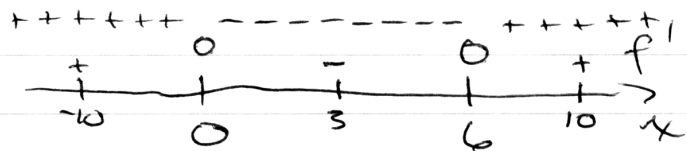
f' is continuous everywhere

f' is 0 for $x = 0$ or $x = 6$

$$f'(-10) = 480 > 0$$

$$f'(3) = -27 < 0$$

$$f'(10) = 120 > 0$$



f is inc on $(-\infty, 0)$ and on $(6, \infty)$

f is dec on $(0, 6)$

f is neither inc nor dec for $x = 0$ or $x = 6$.

$$(10) f(x) = \frac{x}{1+x^2}$$

(f is cont. & diff everywhere since $1+x^2 > 0$ everywhere; f is the ratio of two polynomials)

$$f'(x) = \frac{(1)(1+x^2) - x(2x)}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

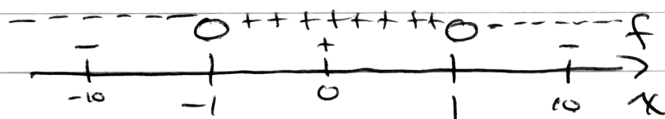
f' is continuous everywhere ($(1+x^2)^2 > 0$ everywhere)

f' is 0 at $x = \pm 1$ (a ratio is 0 only if its numerator is 0 and its denominator isn't)

$$f'(-10) = -\frac{99}{101^2} < 0$$

$$f'(0) = 1 > 0$$

$$f'(10) = -\frac{99}{101^2} < 0$$



f is incr on $(-1, 1)$

f is decr on $(-\infty, -1)$ and on $(1, \infty)$

f is neither incr nor decr at ± 1

$$(22) f(x) = x + \sin x$$

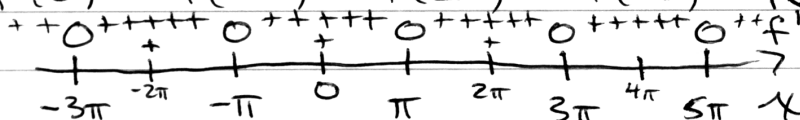
$$f'(x) = 1 + \cos x$$

NB: $\cos x \geq -1$ for all x

f' is cont. everywhere

f' is 0 for $x = \pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \dots$

$$f'(0) = f'(2\pi) = f'(4\pi) = f'(6\pi) = \dots = 2 > 0$$



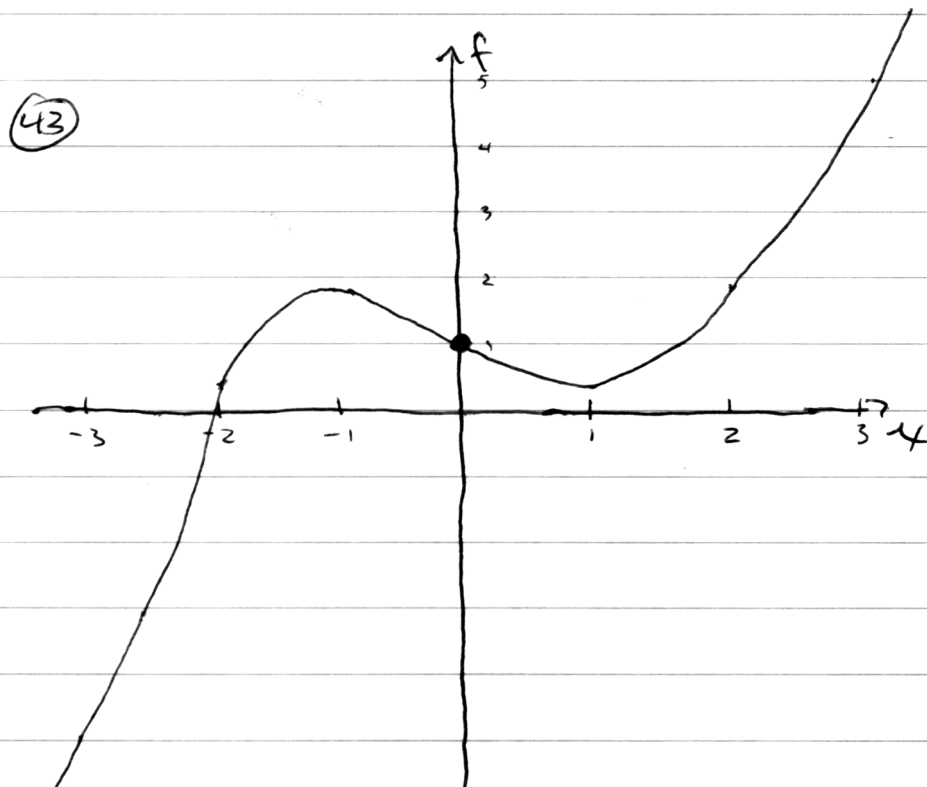
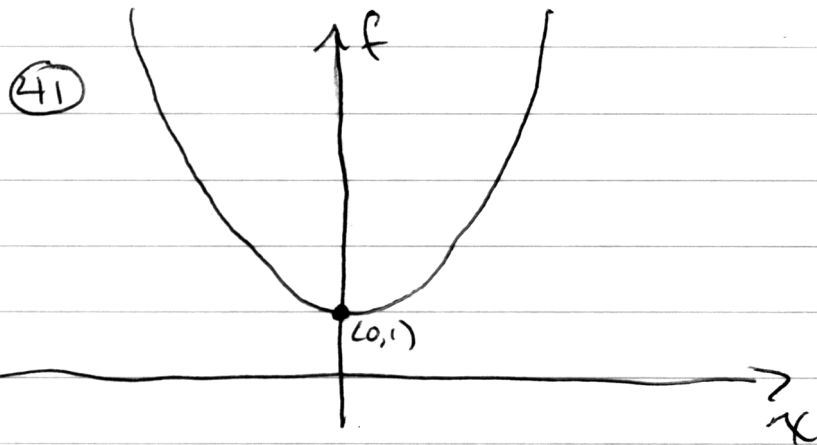
f is incre on $\dots, (-5\pi, -3\pi), (-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi), (3\pi, 5\pi), (5\pi, 7\pi), (7\pi, 9\pi), \dots$

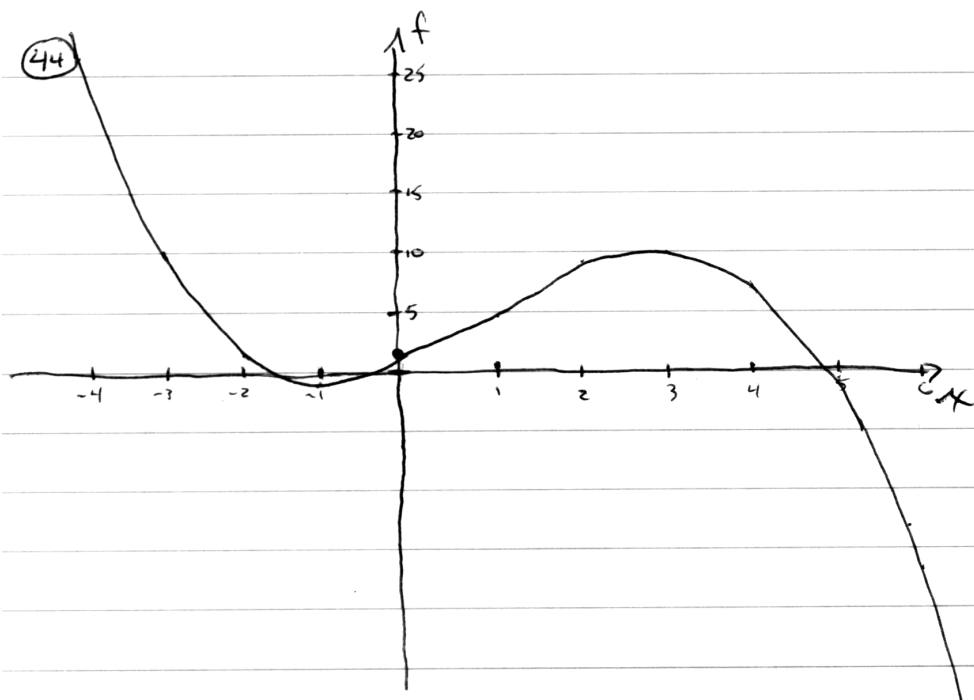
(That is, everywhere but where it is 0)

f is never decr.; f is neither incr nor decr. at $\pm\pi, \pm 3\pi, \dots$

28) $f(x) = \frac{x^3}{3} - x + 1$

30) $f(x) = x^2 - 5x + 10$





4.3 Find the critical #s of f & the local extreme values

① $f(x) = x^3 + 3x - 2$

f is cont. & diff. everywhere so there are no crit. values of f where f' doesn't exist. The only critical values are where $f'(x) = 3x^2 + 3$ is 0. However, f' is never 0 so f has no critical values and no local maxes or mins.

② $f(x) = 2x^4 - 4x^2 + 6$

f is cont. & diff. everywhere.

$f'(x) = 0$ where $8x^3 - 8x = 0$. That is where

$x = -1, x = 0, \text{ or } x = 1$ (think $8x^3 - 8x = 8x(x-1)(x+1)$)

Since $f''(x) = 24x^2 - 8$ and $f''(-1) = 16$, $x = -1$ is a local min. Since $f''(0) = -8$, $x = 0$ is a local max. Since $f''(1) = 16$, $x = 1$ is a local min.

③ $f(x) = x^2 - \frac{3}{x^2}$

f is cont. & diff. everywhere but $x = 0$. Thus $x = 0$ is a critical value (different people define "critical value" in different ways; ultimately, our goal is to identify values

④ (continued) where the function — and its derivative — do "weird" things; $x=0$ is definitely one such place for this function). $f'(x) = 2x + \frac{6}{x^3}$ is never 0. (When $x > 0$, $2x > 0$ and $\frac{6}{x^3} > 0$. Thus $2x + \frac{6}{x^3} > 0$. When $x < 0$, $2x < 0$ and $\frac{6}{x^3} < 0$. Thus $2x + \frac{6}{x^3} < 0$.) Thus we have only one critical value at $x=0$. However, f has neither a local max nor a local min at $x=0$ because $f(0)$ is not defined. (A local max/min is a value for which the function at that value is greater/less than the function at places near that value; since f is not defined, we have no basis for comparison. ~~Alternatively~~ Alternatively, note that $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \pm\infty$.)

⑧ $f(x) = \frac{2-3x}{2+x}$

f is cont. & diff. everywhere but at $x=-2$. Thus $x=-2$ is a crit. value.

$$f'(x) = \frac{(-3)(2+x) - (1)(2-3x)}{(2+x)^2} = \frac{-8}{(2+x)^2} \text{ is never } 0.$$

Thus the only crit. value is at $x=-2$. f has neither a local max nor min at $x=-2$, though, because f is not defined at $x=-2$.

②② $f(x) = x^{7/3} - 7x^{1/3}$

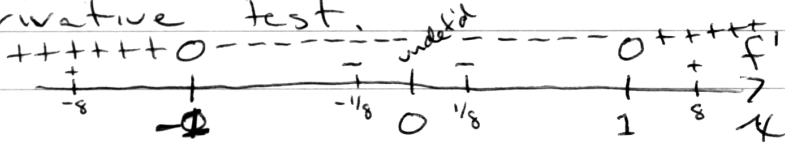
f is cont. everywhere & diff. everywhere but at $x=0$. Thus $x=0$ is a critical value.

$$f'(x) = \frac{7}{3}x^{4/3} - \frac{7}{3}x^{-2/3} \text{ is } 0 \text{ for } x = \pm 1$$

We have three critical values total: $x=-1$, $x=0$, and $x=+1$. Since $f''(x) = \frac{28}{9}x^{1/3} + \frac{14}{9}x^{-5/3}$ and

②② (continued) $f''(-1) = -\frac{28}{9} - \frac{14}{9} = -\frac{14}{3} < 0$, ~~now~~ f has a local ~~max~~ at $x=-1$. Similarly, $f''(1) = \frac{28}{9} + \frac{14}{9} = \frac{14}{3} > 0$ so that f has a local min at $x=1$.

As for $x=0$, $f''(0)$ is undefined so the second derivative test tells us nothing. (Also, $f(0)$ is defined - it's 0 - so we do have a basis for comparison.) However, we can form a sign chart for f' , and use the first derivative test.



$$f'(-\frac{1}{8}) = \frac{7}{3} \cdot \frac{1}{16} - \frac{7}{3} \cdot \frac{1}{4} = \frac{7 \cdot 63}{3 \cdot 4} > 0$$

$$f'(-\frac{1}{8}) = \frac{7}{3} \cdot \frac{1}{16} - \frac{7}{3} \cdot \frac{1}{4} = -\frac{7 \cdot 63}{3 \cdot 16} < 0$$

$$f'(\frac{1}{8}) = \frac{7}{3} \cdot \frac{1}{16} - \frac{7}{3} \cdot \frac{1}{4} = -\frac{7 \cdot 63}{3 \cdot 16} < 0$$

$$f'(\frac{7}{8}) = \frac{7}{3} \cdot \frac{1}{4} - \frac{7}{3} \cdot \frac{1}{4} = \frac{7 \cdot 63}{3 \cdot 4} > 0$$

Thus there is neither a max nor a min at $x=0$ because f' does not change sign at $x=0$.

(NB: The first derivative test confirms the results of the second derivative test at $x=\pm 1$.)

②⑥ $f(x) = x + \cos(2x)$ for only $0 < x < \pi$

f is cont. & diff. everywhere

$f'(x) = 1 - 2\sin(2x)$ is 0 when:

$$0 = 1 - 2\sin(2x)$$

$$\Rightarrow \sin(2x) = \frac{1}{2}$$

$$\Rightarrow 2x = \dots, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \dots$$

$$\Rightarrow x = \dots, -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \dots$$

Since the problem restricts our attention to only $x \in (0, \pi)$, we need only consider solutions $x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$. From above, since $f''(x) = -4\cos(2x)$ and $f''(\frac{\pi}{12}) = -4 \cdot \frac{\sqrt{3}}{2} = -2\sqrt{3} < 0$, ~~now~~ f has a max at $x = \frac{\pi}{12}$.

② (continued) Similarly, since $f''(5\pi/12) = -4(-\frac{\sqrt{3}}{2}) = 2\sqrt{3} > 0$,
f has a min at $x = 5\pi/12$.

③① $f(x) = 2\sin^3 x - 3\sin x$ for only $0 < x < \pi$

f is cont. & diff. everywhere

$f'(x) = (6\sin^2 x)(\cos x) - 3\cos x$ is 0 when:

$$0 = 6\sin^2 x \cos x - 3\cos x$$

$$\Rightarrow 0 = (3\cos x)(2\sin^2 x - 1)$$

$$\begin{aligned} \Rightarrow 3\cos x = 0 & \Rightarrow \cos x = 0 \Rightarrow x = \dots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots \\ \text{or } 2\sin^2 x - 1 = 0 & \Rightarrow \sin x = \frac{1}{\sqrt{2}} \Rightarrow x = \dots, \frac{7\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \dots \\ & \Rightarrow \sin x = -\frac{1}{\sqrt{2}} \Rightarrow x = \dots, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \dots \end{aligned}$$

Since the problem restricts our attention to $0 < x < \pi$, we need only consider the critical values $x = \pi/4$, $x = \pi/2$, and $x = 3\pi/4$. Since $f''(x) = (12\sin x \cos x)(\cos x) + (6\sin^2 x)(-\sin x) + 3\sin x$, and $f''(\pi/4) = 12 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - 6(\frac{1}{\sqrt{2}})^2 \cdot \frac{1}{\sqrt{2}} + 3 \cdot \frac{1}{\sqrt{2}} = 6 \cdot \frac{1}{\sqrt{2}} > 0$, then f has a min at $x = \pi/4$.

Since $f''(\pi/2) = 12 \cdot 1 \cdot 0 \cdot 0 - 6 \cdot 1^2 \cdot 1 + 3 \cdot 1 = -3$, then f has a max at $x = \pi/2$. Since $f''(3\pi/4) = 12 \cdot \frac{1}{\sqrt{2}} \cdot (-\frac{1}{\sqrt{2}}) \cdot (-\frac{1}{\sqrt{2}}) - 6(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{\sqrt{2}}) + 3 \cdot \frac{1}{\sqrt{2}} = 6 \cdot \frac{1}{\sqrt{2}} > 0$, then f has a min at $x = 3\pi/4$.