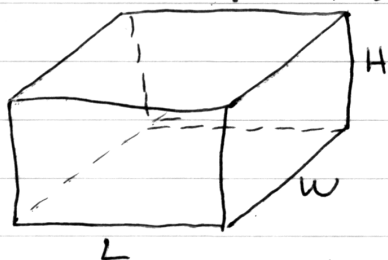


H.S.

- ⑧ What are the dimensions of the base of the rectangular box of greatest volume that can be constructed from  $100\text{in}^2$  of cardboard if the base is to be twice as long as it is wide? Assume that the box has a top.



$$V = LWH$$

↳ this is what we want to maximize

We are also told that we have only  $100\text{in}^2$  of cardboard to use for the box. If  $L, W$ , &  $H$  are measured in inches, then

$$100 = 2LW + 2LH + 2WH$$

We are also told that the base is twice as long as it is wide. That is,  $L = 2W$ . Thus we can eliminate two of the three variables in the expression for volume. I'll pick  $L$  &  $H$  to eliminate (but any two are fine).

Since  $L = 2W$ , then  $100 = 4W^2 + 4WH + 2WH$  so that

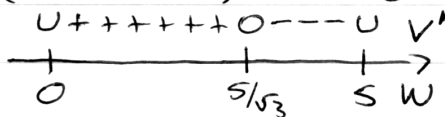
$$H = \frac{50 - 2W^2}{3W}. \quad \text{Thus } V(W) = (2W)(W)\left(\frac{50 - 2W^2}{3W}\right)$$

(note:  $W=0$  is a physical impossibility so we are safe in ignoring it.)

$$= \frac{2}{3}(50W - 2W^3) \quad \text{for } W \in [0, 5]$$

$$\text{and } V'(W) = \frac{2}{3}(50 - 6W^2) \quad \text{for } W \in (0, 5)$$

$V(W)$  has critical points where  $V'(W)$  is undefined ~~at~~ (happens ~~at~~ at the "endpoints" of  $W=0$  &  $W=5$ ) or where  $V'(W) = \frac{2}{3}(50 - 6W^2)$  is 0 (where  $50 = 6W^2$  or  $W = 5/\sqrt{3}$ ).



By the first derivative test,  $W=0$  &  $W=5$  are minima,

⑮ (continued) and  $W = 5/\sqrt{3}$  is a <sup>(local)</sup> maximum. Since  $W = 5/\sqrt{3}$  is the only local maximum and  $V(W)$  is continuous on  $[0, 5]$ , then  $W = 5/\sqrt{3}$  is the global maximum.

Thus the dimensions of the base of the largest box are  $5/\sqrt{3}$  in  $\times 10/\sqrt{3}$  in (the height is  $20/3\sqrt{3}$  in and the volume is  $1000/9\sqrt{3}$  in<sup>3</sup>).

⑯ (same as problem ⑮) but that the box has no top)

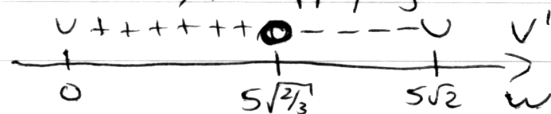
We want to maximize  $V (=LWH)$ . We know that  $L = 2W$  and  $100 = LW + 2LH + 2WH$ .

Thus  $V(W) = (2W)(W)\left(\frac{100-2W^2}{6W}\right)$  for  $W \in [0, 5\sqrt{2}]$

(because  $H = \frac{100-2W^2}{6W}$  and we need  $L \geq 0$ ,  $W \geq 0$ , and  $H \geq 0$ ).

or rather  $V(W) = \frac{1}{3}(100W - 2W^3)$ . Also then  $V'(W) = \frac{1}{3}(100 - 6W^2)$  for  $W \in (0, 5\sqrt{2})$ .

$V(W)$  has critical points at  $W=0$  &  $W=5\sqrt{2}$  (where  $V'(W)$  is undefined) and at  $W = 5\sqrt{2/3}$  (where  $100 = 6W^2$  from  $V'(W)=0$ ). Applying the first derivative test:



$V$  has a minimum at  $W=0$ ,  
a minimum at  $W=5\sqrt{2}$ ,

and a maximum at  $5\sqrt{2/3} = W$ . Since  $V$  is continuous on  $[0, 5\sqrt{2}]$  and the only max is at  $W = 5\sqrt{2/3}$ , then the global max occurs at  $W = 5\sqrt{2/3}$ .

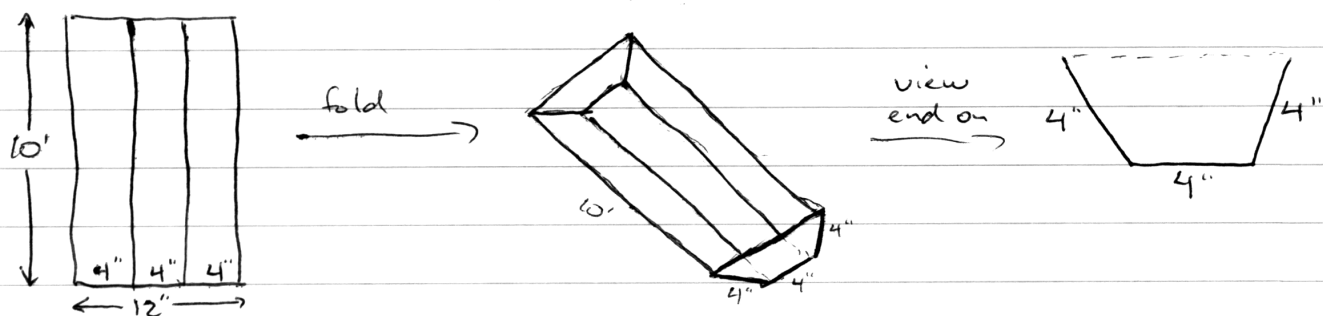
Thus the dimensions of the base of the box with

(16) (continued) greatest volume are  $5\sqrt[2]{3}$  in  $\times$   $10\sqrt[2]{3}$  in (the height is  $\frac{20}{3\sqrt{6}}$  in and the volume is  $\frac{2000}{9\sqrt{6}}$  in<sup>3</sup>).

Note: This is a bigger box by a factor of  $\sqrt{2}$  over the box in (15) (that is, this box is more than 40% larger).

(22) A 10 ft section of gutter is made from a 12 in wide strip of sheet metal by folding up 4 in strips on each side so that they make the same angle with the bottom of the gutter. Determine the depth of the gutter that has the greatest carrying capacity.

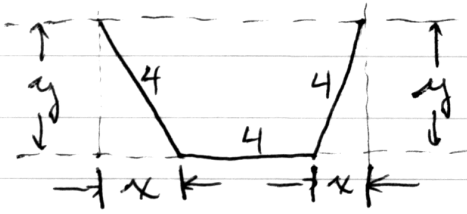
By "carrying capacity" I'm presuming that they're referring to the volume of water that the gutter would hold if we capped the ends. We want to find the volume, then, of a trapezoidal prism:



The volume of a prism is the area of its base times its length. The "base" here is trapezoidal; the area of a trapezoid is the average of its top & bottom widths times its height.

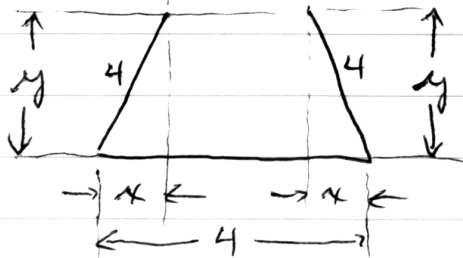
Since the length is fixed at 10 ft (120 in) and the only thing that changes is the "base", let's just

②② (continued) focus on the picture of the base:



The area of the region enclosed by the trapezoid is  $\frac{1}{2}((4) + (4 + 2x))(y)$ .

As the problem notes, we can fold the sides past the vertical. This results in a configuration like that below. If we still label as before but think of  $x$  as negative, our formula for the area is still correct.



$$\frac{1}{2}((4) + (4 + 2x))(y)$$

despite the addition here, if  $x < 0$  then  $4 + 2x < 4$ .

$$\begin{aligned} \text{Thus } V_{\text{gutter}} &= (120) \left( \frac{1}{2} (4 + 4 + 2x)(y) \right) \\ &= 60y(8 + 2x) \end{aligned}$$

We want to maximize  $V$  so we're inclined to differentiate and find critical points. We still have two "independent" variables  $x$  &  $y$ , though. They are related: the right triangle evident above means that  $x^2 + y^2 = 4^2$ , or that  $y = \sqrt{4^2 - x^2}$ .

Here I'm specifically choosing to simplify for  $y$  because we always want  $y \geq 0$  (something I left out above; we want to fold the tabs up). Thus we can ignore the negative root for  $y$ . On the other hand, our physical problem allows  $x$  to range from

(22) (continued) +4 — a flat sheet — all the way down to -2 — when the tips of the two folds meet.

That  $x$  might be negative means when solving for  $x$  in terms of  $y$  ( $x = \pm \sqrt{4^2 - y^2}$ ) we'd need to keep both roots. This is possible to work through, but it gets messy.

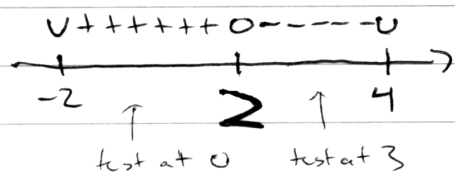
Our volume is then  $V(x) = 60(8+2x)(4^2-x^2)^{1/2}$  for  $x \in [-2, 4]$ .  $V'(x) = 60 \left[ (2)(16-x^2)^{1/2} + (8+2x) \frac{1}{2} (16-x^2)^{-1/2} (-2x) \right]$  for  $x \in (-2, 4)$ . Thus  $V$  has critical points at  $x = -2$  and  $x = 4$  (where  $V'$  is undefined) and for any  $x^*$  that makes  $60(16-x^2)^{1/2} \left[ (2)(16-x^2) - (x)(8+2x) \right] = 0$ .  
 $\underbrace{\quad}_{\text{never } 0} \underbrace{\quad}_{\text{never } 0} \quad 0 \text{ when } 2(16-x^2) = x(8+2x)$

$$\Rightarrow 4x^2 + 8x - 32 = 0$$

$$\Rightarrow x^2 + 2x - 8 = 0$$

$$\Rightarrow (x-2)(x+4) = 0$$

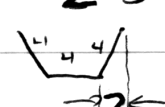
$$\Rightarrow x = \cancel{-4} 2 \quad (x = -4 \text{ is out of range})$$



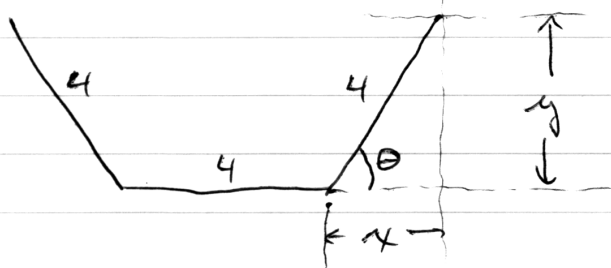
$V(x)$  is continuous for  $x \in [-2, 4]$

By the first derivative test,  $V$  has local mins at  $x = -2$  and  $x = 4$ ,

and  $V$  has a local max at  $x = 2$ . Thus  $V$  has a global max at  $x = \cancel{-4} 2 \cancel{0} \cancel{0} \cancel{0}$

If  $x = 2$ , then  $y = 2\sqrt{3}$ . Thus the depth of the gutter with maximal "volume" is  $2\sqrt{3}$  in (its volume is  $144\sqrt{3}$  in<sup>3</sup> and has a  shape).

② (continued) There are other ways of labelling the above that yield the same answer, but all follow essentially the same path. There is another way to do things that avoids the square root:



This looks like before, but instead of expressing the volume in terms of  $x$ , we do it in terms of  $\theta$ .

We had  $V = 60y(8+2x)$ . Since  $y = 4\sin\theta$  and  $x = 4\cos\theta$ , then

$$V(\theta) = 60(4\sin\theta)(8+2(4\cos\theta)) \quad \text{with } 0 \leq \theta \leq \frac{2\pi}{3}$$

Thus

$$V'(\theta) = 60 \left[ (4\cos\theta)(8+8\cos\theta) + (4\sin\theta)(-8\sin\theta) \right]$$

$$= 60 (32\cos\theta + 32\cos^2\theta - 32\sin^2\theta) \quad \text{for } 0 < \theta < \frac{2\pi}{3}$$

We have critical points at  $\theta = 0$  &  $\theta = \frac{2\pi}{3}$  (where  $V'$  is undefined - nowhere else) and where  $V'(\theta) = 0$ .

Thus we want: (cancelling the 60 & the 32)

$$\cos\theta + \cos^2\theta - \sin^2\theta = 0 \quad \text{path ②}$$

path ①

$$\Rightarrow \cos\theta + \cos(2\theta) = 0$$

$$\Rightarrow 2\cos(\frac{1}{2}\theta)\cos(\frac{3}{2}\theta) = 0$$

(next pg)

$\Rightarrow$

$$\cos\theta + \cos^2\theta - (1 - \cos^2\theta) = 0$$

$$\Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$$

$$\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$$

(next pg)

22 (continued)

$$2 \cos(\frac{1}{2}\theta) \cos(\frac{3}{2}\theta) = 0$$

$$\Rightarrow \text{or } \cos(\frac{1}{2}\theta) = 0$$

$$\cos(\frac{3}{2}\theta) = 0$$

$$\Rightarrow \text{or } \frac{1}{2}\theta = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$$

$$\frac{3}{2}\theta = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$$

$$\Rightarrow \theta = \pm \pi/3, \pm \pi, \pm 5\pi/3, \pm 3\pi, \pm 7\pi/3, \dots$$

$$(2 \cos \theta - 1)(2 \cos \theta + 1) = 0$$

$$\Rightarrow \text{or } 2 \cos \theta - 1 = 0$$

$$2 \cos \theta + 1 = 0$$

$$\Rightarrow \text{or } \cos \theta = 1/2$$

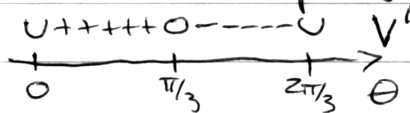
$$\cos \theta = -1$$

$$\Rightarrow \theta = \pm \pi/3, \pm 5\pi/3, \pm 7\pi/3, \dots$$

$$\theta = \pm \pi, \pm 3\pi, \pm 5\pi, \dots$$

$$\Rightarrow \theta = \pm \pi/3, \pm \pi, \pm 5\pi/3, \pm 3\pi, \pm 7\pi/3, \dots$$

Both of these methods (along with any other correct ones you find) give the same answers. Since we are restricted to  $\theta \in (0, 2\pi/3)$ , we need only consider  $\theta = \pi/3$ . ~~Thus~~ Thus our critical points are  $\theta = 0$ ,  $\theta = \pi/3$ , and  $\theta = 2\pi/3$ .



Since  $V$  is continuous on  $[0, 2\pi/3]$ ,

and since  $\theta = 0$  &  $\theta = 2\pi/3$  are local

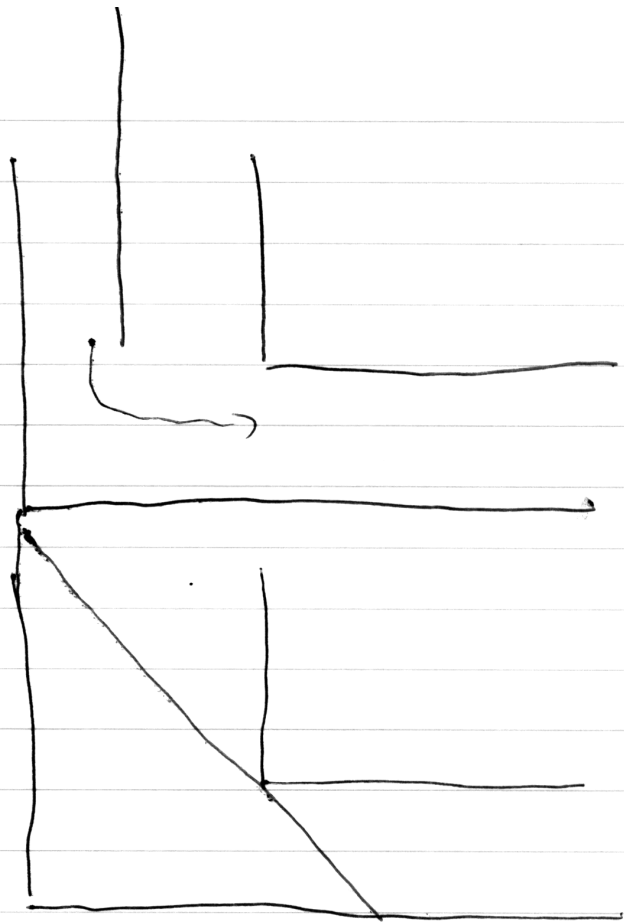
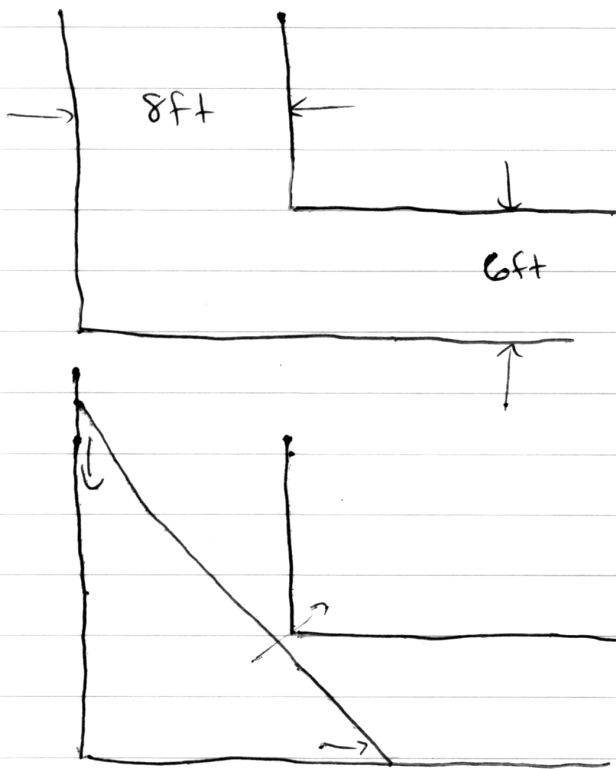
mins and  $\theta = \pi/3$  a local max by the first derivative test, then  $V$  has a global max at  $\theta = \pi/3$ . The depth of the trough,  $y$ , is  $4 \cos(\pi/3) = 4(1/2) = 2$  at  $\theta = \pi/3$ .

This is the depth at which maximum volume is achieved.

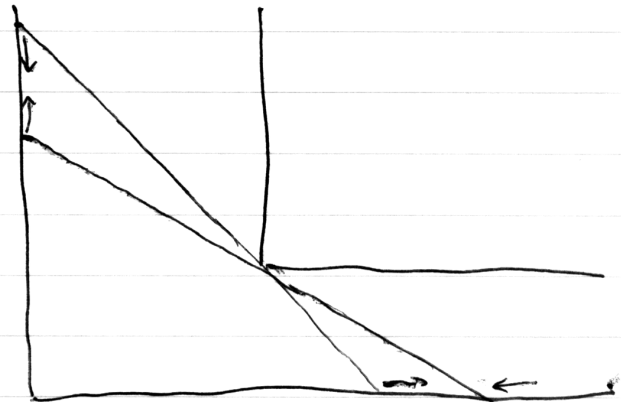
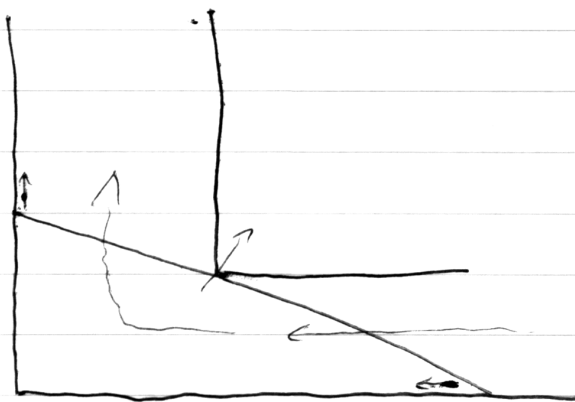
28 ~~Two~~ Two hallways, one 8 ft wide and the other 6 ft wide, meet at right angles. Determine the length of the longest ladder that can be carried horizontally from one hallway into the other.

The set-up for this problem requires some careful reasoning. Considering the picture from the next page, if we carry a very long ladder down the 8-foot-wide corridor, and we try to round the corner the ladder will get wedged against

28 (continued)

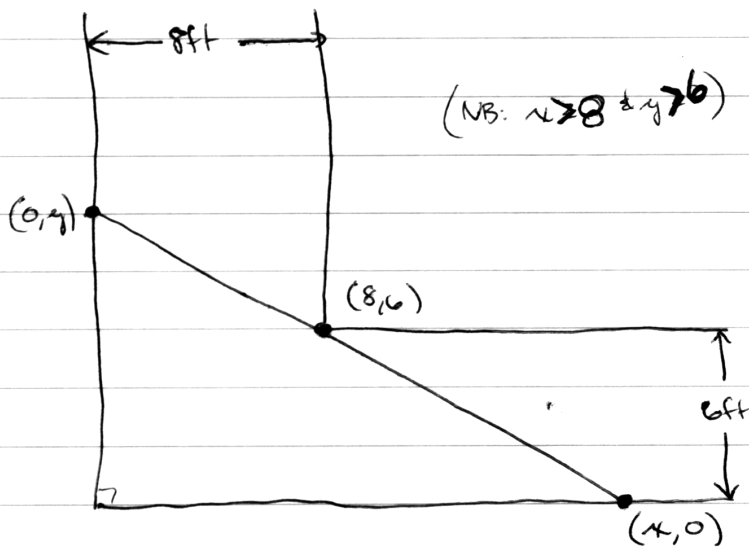


the two outside walls and the inside corner.  
 We can always back the ladder out by sliding the top point along the left outer wall back the way we came; The point of the ladder along the bottom outer wall will lift off. A similar thing happens if we approach from the right through the 6 ft corridor with the same-length long ladder.





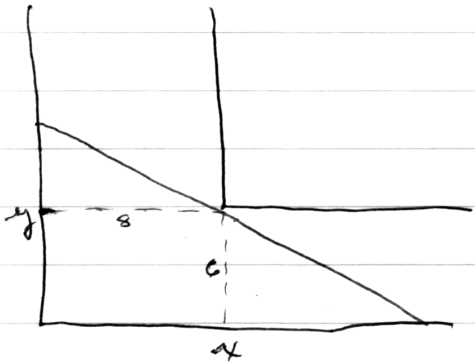
(28) (continued) From the two approaches, the ladder gets wedged into two different positions. If we shorten the ladder until it just barely wedges, these two wedged positions from the two approaches become one and the same position. Since we can always back a ladder out from a wedged position, and since the shortest long ladder gets wedged in the same position from either direction, the shortest long/wedgeable ladder can make it around the corner. (Any longer ladder can't because of the asymmetry; any shorter ladder can make it through with room to spare. ~~Thus~~ This shortest long ladder is the longest ladder which can make it around the corner.)



Any long ladder, when wedged, has three points of contact with the walls. Pretend the left & bottom <sup>outer</sup> walls are a set of Cartesian axes. Then the three contact points are at  $(x, 0)$ ,  $(8, 6)$ , and  $(0, y)$  (measured in feet). The

length of the ladder is going to be  $L = \sqrt{x^2 + y^2}$  by the Pythagorean theorem. Since we want to find the shortest such long ladder, we want to minimize  $L$ . To minimize  $L$  we want to differentiate

(28) (continued) and find critical points. To do that, we need to express  $L$  in terms of a single variable.



There are three similar triangles hiding out in the picture (two small and one large). We can use the fact that corresponding sides of similar triangles to get that:

$$\frac{x-8}{8} = \frac{6}{y-6}, \text{ for instance. There}$$

are many such proportions one could use. They all lead to  $y = \frac{6x}{x-8}$ . (NB:  $\frac{x}{y} \neq \frac{8}{6}$  You need to compare

the sides from two triangles only, not three.) This gives

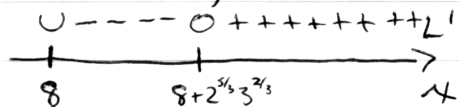
$$L(x) = \sqrt{x^2 + \left(\frac{6x}{x-8}\right)^2} \text{ where } 8 < x < +\infty$$

To minimize  $L$ , we find critical points: those  $x$ 's that make  $L'$  undefined (there are none in  $(8, \infty)$ ), and those  $x$ 's that make  $L'(x) = 0$ . To find the second kind:

$$L'(x) = \frac{1}{2} \left( x^2 + \left( \frac{6x}{x-8} \right)^2 \right)^{-1/2} \left( 2x + 2 \left( \frac{6x}{x-8} \right) \left( \frac{6(x-8) - (6x)(1)}{(x-8)^2} \right) \right)$$

$$(\text{since } x > 8) = \left( 1 + \left( \frac{6}{x-8} \right)^2 \right)^{-1/2} \left( 1 + \left( \frac{6}{x-8} \right) \left( -\frac{48}{(x-8)^2} \right) \right)$$

$L'(x) = 0$  when  $1 - \frac{288}{(x-8)^3} = 0$  (NB:  $\left( 1 + \left( \frac{6}{x-8} \right)^2 \right)^{-1/2}$  is never 0). We want  $(x-8)^3 = 288 = 2^5 \cdot 3^2 \Rightarrow x = 8 + 2^{5/3} 3^{2/3}$



Thus by the first derivative test,  $L$  has a local min at  $x = 8 + 2^{5/3} 3^{2/3}$

Since  $L$  is continuous on  $(8, \infty)$  and this is the only local min, it must be the global min.

28 (continued) when  $x = 8 + 2^{5/3} 3^{2/3}$ , we can get  $y$  from  $\frac{(8 + 2^{5/3} 3^{2/3}) - 8}{8} = \frac{6}{y-6} \Rightarrow y-6 = \frac{6 \cdot 8}{2^{5/3} 3^{2/3}} \Rightarrow y = 6 + 2^{7/3} 3^{1/3}$ .

Thus the longest ladder which can fit around the corner (which was the shortest "long" ladder) is of length  $L = \sqrt{(8 + 2^{5/3} 3^{2/3})^2 + (6 + 2^{7/3} 3^{1/3})^2}$

$$\begin{aligned}
 &= \left( 2^6 + 2^{17/3} 3^{2/3} + 2^{10/3} 3^{4/3} + 2^2 3^2 + 2^{13/3} 3^{4/3} + 2^{14/3} 3^{2/3} \right)^{1/2} \\
 &= 2 \left( 2^4 + 2^{11/3} 3^{2/3} + 2^{4/3} 3^{4/3} + 3^2 + 2^{7/3} 3^{4/3} + 2^{8/3} 3^{2/3} \right)^{1/2} \\
 &= 2 \left[ (2^{4/3} + 3^{2/3})(2^{8/3} + 3^{4/3} + 2^{7/3} 3^{2/3}) \right]^{1/2} \\
 &= 2 \left[ (2^{4/3} + 3^{2/3})(2^{4/3} + 3^{2/3})^2 \right]^{1/2} \\
 &= 2 (2^{4/3} + 3^{2/3})^{3/2}
 \end{aligned}$$

There are other ways to write  $L$  in terms of a single variable. For instance, one could use the "slope" of the ladder in the Cartesian coordinate system. Or, like I did in 22, one could use the angle at which the ladder meets the  $x$ -axis. Though the algebra may be different in each case, you should still find the same ladder length.

One could also consider "T" or "+" hallway intersections in a different interpretation. Although there are more possibilities for moving the ladder around, this doesn't allow one to get any longer a ladder through (Why? :)